Algebra

Initialization

In[1]:=
```
ClearAll["Global`*"];
Off[General::spell, General::spell1]
```

In[3]:=
```
Needs["Notation`"]
```

Manipulation of algebraic expressions

Rational expressions

Together collects terms over a common denominator and cancels common factors.

In[4]:= $\frac{a}{b+c} - \frac{d}{e-f}$ // Together

Out[4]= $\frac{-b\,d - c\,d + a\,e - a\,f}{(b+c)\,(e-f)}$

Apart separates an expression into a sum of terms with simpler denominators.

In[5]:=
```
% // Apart
```

Out[5]=
$$\frac{a}{b+c}+\frac{d}{-e+f}$$

Cancel cancels common factors between the numerator and denominator of a rational expression.

In[6]:=
$$\frac{2\,a^2\,b}{(b+c)\,a}\ //\ \text{Cancel}$$

Out[6]=
$$\frac{2\,a\,b}{b+c}$$

The Numerator or Denominator of a rational function of polynomials can be extracted using functions of the same name. Note that the numerator is considered the part of the expression which does not have a "superficially" negative power.

In[7]:=
$$\text{expr} = \frac{a\,b^{-2}\,g^{-n}}{(c-d)\,(e-f)^{-1}};$$

```
{Numerator[expr], Denominator[expr]}
```

Out[8]=
$$\left\{a\,(e-f),\ b^2\,(c-d)\,g^n\right\}$$

Note that the option **Trig→True** is needed for use with trigonometric expressions to avoid automatic conversions of expressions of the type 1/Cos[*x*]→Sec[*x*].

In[9]:=

```
expr = Sin[x^2]/Cos[x^3];
{Numerator[expr], Denominator[expr],
  Denominator[expr, Trig → True]}
```

Out[10]=

$$\{\mathrm{Sec}[x^3]\,\mathrm{Sin}[x^2],\ 1,\ \mathrm{Cos}[x^3]\}$$

Combine: $\frac{x}{1-x^2} - \frac{1}{1+x}$

Expanding products

ExpandNumerator expands numerators while ExpandDenominator expands denominators. (Duh!)

In[11]:=

$$\frac{(a+b)(b-2\,a)}{(a-b)\,a}\ //\ \mathrm{ExpandNumerator}$$

Out[11]=

$$\frac{-2\,a^2-a\,b+b^2}{a\,(a-b)}$$

In[12]:=

$$\frac{(a+b)(b-2\,a)}{(a-b)\,a}\ //\ \mathrm{ExpandDenominator}$$

Out[12]=

$$\frac{(-2\,a+b)(a+b)}{a^2-a\,b}$$

Expand expands numerators leaving denominators in factored form, cancelling common factors where possible.

In[13]:=

$$\frac{(a+b)(b-2a)}{(a-b)a} \text{ // Expand}$$

Out[13]=

$$-\frac{2a}{a-b}-\frac{b}{a-b}+\frac{b^2}{a(a-b)}$$

ExpandAll expands both numerators and denominators. Note that common factors are not cancelled.

In[14]:=

$$\frac{(a+b)(b-2a)}{(a-b)a} \text{ // ExpandAll}$$

Out[14]=

$$-\frac{2a^2}{a^2-ab}-\frac{ab}{a^2-ab}+\frac{b^2}{a^2-ab}$$

Cancel common factors from the expression above.

Factoring sums

Factor reduces an expression to a product of factors.

In[15]:=

$$(a+b)(2a-b)(c-a)(d+b) \text{ // Expand}$$

Out[15]=

$$-2a^3b-a^2b^2+ab^3+2a^2bc+ab^2c-b^3c-2a^3d-a^2bd+ab^2d+2a^2cd+abcd-b^2cd$$

In[16]:=

```
% // Factor
```

Out[16]=

$$-(2a-b)(a+b)(a-c)(b+d)$$

In[17]:=

$$\frac{(a+b)(2a-b)}{(c-a)(d+b)} \text{ // ExpandAll}$$

Out[17]=

$$\frac{2a^2}{-ab+bc-ad+cd}+\frac{ab}{-ab+bc-ad+cd}-\frac{b^2}{-ab+bc-ad+cd}$$

In[18]:=

```
% // Factor
```

Out[18]=

$$-\frac{(2a-b)(a+b)}{(a-c)(b+d)}$$

By default, **Factor** is limited to integers and thus will not factor

In[19]:=

$$x^2+1 \text{ // Factor}$$

Out[19]=

$$1+x^2$$

The option **GaussianIntegers→True** can be used to extend **Factor** to complex numbers.

In[20]:=

```
Factor[x^2 + 1, GaussianIntegers → True]
```

Out[20]=

```
(-ⅈ + x) (ⅈ + x)
```

In[21]:=

```
Options[Factor]
```

Out[21]=

```
{Extension → None, GaussianIntegers → False,
  Modulus → 0, Trig → False}
```

Similarly, with the option **Trig→True** multiple-angle formulae are used to factor expressions involving trigonometric and hyperbolic functions.

In[22]:=

```
Factor[Sin[x] Sin[3 x], Trig → True]
```

Out[22]=

```
(1 + 2 Cos[2 x]) Sin[x]^2
```

In[23]:=

```
Factor[Sinh[2 x] Cosh[2 x], Trig → True]
```

Out[23]=

```
2 Cosh[x] Cosh[2 x] Sinh[x]
```

Factor : $-24\,x + 12\,x^2 - 24\,y + 4\,x\,y + 10\,x^2\,y - 3\,x^3\,y - 8\,y^2 + 10\,x\,y^2 - x^2\,y^2 - x^3\,y^2 + 2\,x\,y^3 - x^2\,y^3$

Factor:

a) Cos[x]+Cos[y] b) Cosh[x]+Cosh[y]

Collecting terms

Collect**[expr,x]** collects terms involving the same powers of **x**.

In[24]:=

```
(a + b x)(c x - d x^2) // Collect[#, x] &
```

Out[24]=

$$a c x + (b c - a d) x^2 - b d x^3$$

Collect[expr,x,f] applies the function **f** to each coefficient separately. Thus, **Collect[expr,x,Simplify]** can be used to simplify the coefficients or **Collect[expr,x,Factor]** to factor them.

In[25]:=

```
(a + b x)(a c x - d x^2) // Collect[#, x, Factor] &
```

Out[25]=

$$a^2 c x + a (b c - d) x^2 - b d x^3$$

Collect[expr,{x1,x2}] collects terms involving the same powers from a list of objects.

In[26]:=

```
(a y + b x + f)(c y^2 - d x^2 - g) // Collect[#, {x, y}] &
```

Out[26]=

$$-f g - b d x^3 - a g y + c f y^2 + a c y^3 + x^2 (-d f - a d y) + x \left(-b g + b c y^2\right)$$

In[27]:=

```
(a y + b x + f) (c y^2 - d x^2 - g) // Collect[#, {y, x}] &
```

Out[27]=

$$-f g - b g x - d f x^2 - b d x^3 + \left(-a g - a d x^2\right) y + (c f + b c x) y^2 + a c y^3$$

Note that collection priority is ordered according to the listed order of the variables.

FactorTerms**[expr,x]** extracts any that are independent of **x**.

In[28]:=

```
-a d x - b d y + a c x y + b c y^2 // FactorTerms[#, x] &
```

Out[28]=

$$-(a x + b y) (d - c y)$$

In[29]:=

```
-a d x - b d y + a c x y + b c y^2 // FactorTerms[#, {x, y}] &
```

Out[29]=

$$(a x + b y) (-d + c y)$$

With only one argument an overall rational factor is pulled out of a polynomial.

In[30]:=

$$2 x + \frac{1}{2} + \frac{x^2}{3} \text{ // FactorTerms}$$

Out[30]=

$$\frac{1}{6} \left(3 + 12 x + 2 x^2\right)$$

Coefficients are obtained using Coefficient or CoefficientList. **Coefficient[expr,x]** reports the coefficient of **x** found in expression **expr**; note that for this purpose x^2 is considered distinct from x, such that the coefficient returned will not contain any power of x. **Coefficient[expr,x,n]** returns the coefficient of x^n.

In[31]:=

$$\text{expr} = \left(a + 3\,x\,y + \frac{b}{x}\right)^{15};$$
$$\{\text{Coefficient}[\text{expr}, x^7], \text{Coefficient}[\text{expr}, x^2, 7], \text{Coefficient}[\text{expr}, x, -3]\}$$

Out[32]=

$$\{14\,073\,345\,a^8\,y^7 + 295\,540\,245\,a^6\,b\,y^8 + 1\,477\,701\,225\,a^4\,b^2\,y^9 + 1\,773\,241\,470\,a^2\,b^3\,y^{10} + 241\,805\,655\,b^4\,y^{11},\ 71\,744\,535\,a\,y^{14},\ 455\,a^{12}\,b^3 + 45\,045\,a^{10}\,b^4\,y + 1\,216\,215\,a^8\,b^5\,y^2 + 11\,351\,340\,a^6\,b^6\,y^3 + 36\,486\,450\,a^4\,b^7\,y^4 + 32\,837\,805\,a^2\,b^8\,y^5 + 3\,648\,645\,b^9\,y^6\}$$

CoefficientList[expr,x] returns a list of coefficients beginning with x^0. **Coefficient[expr,{x1,x2,...}]** returns a rectangular array. Note that since both **Coefficient** and **CoefficientList** are both intended for use primarily with polynomials, functions other than powers may appear in the coefficients. Also, **CoefficientList** does not handle negative powers.

In[33]:=

```
expr = (x + y Cos[y])^5;
CoefficientList[expr, x]
```

Out[34]=

$\{y^5 \text{Cos}[y]^5, 5 y^4 \text{Cos}[y]^4, 10 y^3 \text{Cos}[y]^3, 10 y^2 \text{Cos}[y]^2, 5 y \text{Cos}[y], 1\}$

In[35]:=

```
CoefficientList[(x + y)^5, {x, y}] // MatrixForm
```

Out[35]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Let expr = Nest[μ #(1-#)&,x,4].
a) Use Collect to produce an explicit polynomial in μ and to factor the coefficients.
b) Alternatively, use CoefficientList to list the coefficients of μ^n and then factor those coefficients.

Deduce the linear transformation $x \to a z + b$ that takes an arbitrary quadratic function $\alpha x^2 + \beta x + \gamma$ to the simpler form $z^2 + c$ by determining $\{a, b, c\}$ in terms of $\{\alpha, \beta, \gamma\}$.

Using the AlgebraicManipulation palette

There exists an **AlgebraicManipulation** palette which can be used to perform many of the operations by *point-and-click* rather than by entering commands directly. If you are unsure of whether that palette is active, use the Window submenu of the main toolbar to display a list of the active windows. If **AlgebraicManipulation** is present, clicking on it will bring it to the fore; otherwise use File ↦ Palettes to activate it. The palette contains a list of operations that can be applied to a highlighted expression or subexpression. Highlight the following expression or a portion thereof. Then apply one of the transformations, such as **Expand**, by clicking on the palette. That operation is performed *in situ*, leaving the result highlighted. Clicking on **Factor** undoes the previous the operation. **Expand** followed by **Simplify** produces another variation. Experimenting with various transformation sequences can produce the desired result more quickly than typing and editing commands. Of course, if you expect to repeat these evaluations later, you should record the best sequence of transformations in the notebook itself.

In[36]:=

$$-(2a-b)(a+b)(a-c)(b+d)$$

Out[36]=

$$(-2a+b)(a+b)(a-c)(b+d)$$

This method can be applied to parts of expressions also. To get some familiarity with this technique, perform the following operations on a *copy* of the expression below.

1) put the first term in the numerator over a common denominator

2) factor the second term of the numerator

3) simplify the numerator

4) try simplifying the denominator — the result is not likely to be satisfactory

5) try other sequences on the whole or parts of the denominator

In[37]:=

$$\frac{\left(a + b\,\frac{x}{c-d}\right)\left(c^2 - d^2\right)}{a + \frac{b}{c+\frac{d}{x-d}} + \frac{1}{c\,x - d\,c + d}}$$

Out[37]=

$$\frac{\left(c^2 - d^2\right)\left(a + \frac{b\,x}{c-d}\right)}{a + \frac{1}{d - c\,d + c\,x} + \frac{b}{c+\frac{d}{-d+x}}}$$

Manipulation of trigonometric expressions

Many functions are also available for manipulation of expressions involving trigonometric functions, often appearing as counterparts to functions that transform algebraic expressions in similar ways. Both circular and hyperbolic trigonometry are handled by these functions. In addition, there are functions which transform between trigonometric and exponential representations.

TrigExpand expands products of trigonometric functions into a sum of terms and applies multiple-angle formulas to reduce the arguments to their most basic form.

In[38]:=

```
Cos[2 x] Sin[2 x] // TrigExpand
```

Out[38]=

$$2\,\mathrm{Cos}[x]^3\,\mathrm{Sin}[x] - 2\,\mathrm{Cos}[x]\,\mathrm{Sin}[x]^3$$

In[39]:=

```
Cosh[2 x] Sinh[2 x] // TrigExpand
```

Out[39]=

$$2\,\mathrm{Cosh}[x]^3\,\mathrm{Sinh}[x] + 2\,\mathrm{Cosh}[x]\,\mathrm{Sinh}[x]^3$$

TrigFactor factors expressions into products of terms.

In[40]:=

```
2 Cosh[x]^3 Sinh[x] + 2 Cosh[x] Sinh[x]^3 // TrigFactor
```

Out[40]=

$$2\,\text{Cosh}[x]\,(\text{Cosh}[x] - i\,\text{Sinh}[x])\,(\text{Cosh}[x] + i\,\text{Sinh}[x])\,\text{Sinh}[x]$$

TrigReduce uses trigonometric identities to simplify expressions, generally attempting to reduce the number of trigonometric functions involved, often using multiple-angle formulas.

In[41]:=

```
Sin[x]^2 + Cos[x]^2
```

Out[41]=

$$\text{Cos}[x]^2 + \text{Sin}[x]^2$$

In[42]:=

```
Sin[x]^2 + Cos[x]^2 // TrigReduce
```

Out[42]=

1

In[43]:=

```
2 Cos[x]^3 Sin[x] - 2 Cos[x] Sin[x]^3 // TrigReduce
```

Out[43]=

$$\frac{1}{2}\,\text{Sin}[4\,x]$$

TrigToExp converts trigonometric functions to exponential form.

In[44]:=

```
stuff = Cosh[2 x]^2 - Sinh[x] // TrigToExp
```

Out[44]=

$$\frac{1}{2} + \frac{e^{-4x}}{4} + \frac{e^{-x}}{2} - \frac{e^{x}}{2} + \frac{e^{4x}}{4}$$

ExpToTrig converts from exponential to trigonometric functions.

In[45]:=

```
stuff // ExpToTrig
```

Out[45]=

$$\frac{1}{2} + \frac{1}{2}\text{Cosh}[4\,x] - \text{Sinh}[x]$$

Often several transformations can be profitably combined.

In[46]:=

$$\sum_{m=-j}^{j} \text{Exp}\left[\frac{m\,y}{j}\right] \text{ // ExpToTrig // TrigFactor}$$

Out[46]=

$$\text{Csch}\left[\frac{y}{2\,j}\right] \text{Sinh}\left[y + \frac{y}{2\,j}\right]$$

Identities can also be checked, but it will often be necessary to use **`Simplify`** to obtain comparable expressions in both arguments of **`Equal`**.

In[47]:=

$$\text{Cos}[x + y]^2 == \frac{1}{2}(1 + \text{Cos}[2\,x + 2\,y]) \text{ // Simplify}$$

Out[47]=

```
True
```

Expand:
a) Cos[4x] b) Tanh[2x]

Simplify using TrigReduce: $2\cosh(x)^3\sinh(x)+2\cosh(x)\sinh(x)^3$.

Expand and place over common denominator:
a) $\cos(x+y)$ b) $\tanh(x-y)$

Expand the following product and use TrigReduce to simplify the result: $(\cos(x)\cos(y)-\sin(x)\sin(y))^2$.

Use ExpToTrig and TrigFactor to simplify $\frac{1+e^x}{1-e^x}$. Compare with the effect of Simplify.

Watch out for that branch!

Mathematica tends to be more careful about its algebra than the average user. For example, many new users are puzzled by the "failure" of **Simplify** to reduce the following expression

In[48]:=

$\sqrt{a^2}$ // Simplify

Out[48]=

$\sqrt{a^2}$

or **Expand** to alter

In[49]:=

```
Log[a b] // Expand
```

Out[49]=

```
Log[a b]
```

but *Mathematica* does not automatically assume that the positive branch of the square root is desired or that variables are positive or even that they are real. If you are confident that all relevant quantities are positive, PowerExpand can be used to simplify powers and products, expand logarithms, and perform related transformations that are valid for positive quantities. Thus,

In[50]:=

```
√(a^2) // PowerExpand
```

Out[50]=

```
a
```

gives the positive branch of the square root and

In[51]:=

```
Log[a^2 b^3] // PowerExpand
```

Out[51]=

```
2 Log[a] + 3 Log[b]
```

separates the log of a product into the sum of their logs assuming that both factors are positive and extracts their exponents properly. However, although **PowerExpand** can be very useful, you must ensure that all affected expressions satisfy its assumptions. Thus, if $a > b$ and $c > 0$ and all are real, then

In[52]:=

```
Log[(a - b) c] // PowerExpand
```

Out[52]=

```
Log[a - b] + Log[c]
```

is handled correctly, but otherwise you must determine the appropriate branch of the complex logarithm function.

Similarly, ComplexExpand assumes that all variables are real, but not necessarily positive. Thus,

In[53]:=

```
Sqrt[-a^2] // ComplexExpand
```

$\sqrt{-a^2}$ // ComplexExpand

Out[53]=

$i\sqrt{a^2}$

still cannot alter this expression because $\sqrt{a^2}$, though real, could still be $\pm a$. Similarly,

In[54]:=

Log[a b] // ComplexExpand

Out[54]=

$i\,\mathrm{Arg}[a\,b] + \mathrm{Log}\left[\sqrt{a^2}\,\sqrt{b^2}\right]$

involves the argument function **Arg** because the sign of $a\,b$ is not known *a priori*. Nevertheless, **ComplexExpand** is helpful for expressions like

In[55]:=

Sin[x + i y] // ComplexExpand

Out[55]=

Cosh[y] Sin[x] + i Cos[x] Sinh[y]

where explicit real and imaginary parts are expressed in terms of real variables. **ComplexExpand[expr,{x1,x2,…}]** assumes that **{x1,x2,…}** are complex but that all other variables are real.

In[56]:=

```
Exp[z^2 (x + i y)] // ComplexExpand[#, {z}] &
```

Out[56]=

$$e^{-2\,y\,\mathrm{Im}[z]\,\mathrm{Re}[z]+x\left(-\mathrm{Im}[z]^2+\mathrm{Re}[z]^2\right)}\,\mathrm{Cos}\left[2\,x\,\mathrm{Im}[z]\,\mathrm{Re}[z]+y\left(-\mathrm{Im}[z]^2+\mathrm{Re}[z]^2\right)\right]+i\,e^{-2\,y\,\mathrm{Im}[z]\,\mathrm{Re}[z]+x\left(-\mathrm{Im}[z]^2+\mathrm{Re}[z]^2\right)}\,\mathrm{Sin}\left[2\,x\,\mathrm{Im}[z]\,\mathrm{Re}[z]+y\left(-\mathrm{Im}[z]^2+\mathrm{Re}[z]^2\right)\right]$$

Caveat emptor: use PowerExpand and ComplexExpand cautiously! You are responsible for ensuring that all affected expressions satisfy the assumptions made by these functions.

Expand and simplify $\coth(x + i\,y)$ assuming that *x* and *y* are real.

Simplicity is in the eye of the beholder

Perhaps the most useful, but often the most frustrating, *Mathematica* function for manipulation of symbolic expressions is Simplify, which attempts to reduce an expression to a simpler form. **`Simplify`** includes expansion, factorization, and many other algebraic transformations. With the option **`Trig→True`**, which is the default, it applies trigonometric identities also. Thus, because **`Simplify`** includes all of the functions described above, plus others, the first step in simplifying an expression is usually to append a simplification command in postfix notation, namely **`expr//Simplify`**, and seeing what you get.

In the contrived example below this simple operation produces immediate gratification

In[57]:=

```
(4 (Cosh[x]^6 Sinh[x]^2 + 2 Cosh[x]^4 Sinh[x]^4 +
        Cosh[x]^2 Sinh[x]^6)) / (8 + 12 x + 6 x^2 + x^3) // Simplify
```

$$\left(4\left(\mathrm{Cosh}[x]^6\,\mathrm{Sinh}[x]^2+2\,\mathrm{Cosh}[x]^4\,\mathrm{Sinh}[x]^4+\mathrm{Cosh}[x]^2\,\mathrm{Sinh}[x]^6\right)\right)\Big/\left(8+12\,x+6\,x^2+x^3\right)\ //\ \mathrm{Simplify}$$

Out[57]=

$$\frac{\mathrm{Sinh}[4\,x]^2}{4\,(2+x)^3}$$

whereas for this next expression it is useful to convert from exponential to trig before simplifying:

In[58]:=

$$\frac{y^2\,e^y}{(1+e^y)^2}\ //\ \mathrm{ExpToTrig}\ //\ \mathrm{Simplify}$$

Out[58]=

$$\frac{1}{4}\,y^2\,\mathrm{Sech}\left[\frac{y}{2}\right]^2$$

Sometimes when **Simplify** is stumped, FullSimplify can help. **FullSimplify** tries a much wider range of transformations and includes rules for many special functions. Thus, **FullSimplify** is useful for the next expression where **Simplify** is not.

In[59]:=

```
2 n BesselJ[n, x] - x BesselJ[n + 1, x] // FullSimplify
```

Out[59]=

```
x BesselJ[-1 + n, x]
```

However, because **FullSimplify** must test a broader range of possibilities, it can become extremely time consuming and can exhaust the memory of your computer. If you hear the hard disk churning, your expression is probably too complicated to handle without human guidance. It is a good idea to save your work before attempting to apply **FullSimplify** to an unwieldy expression in case it becomes necessary to abort the evaluation, which is not always without risk!

Use FullSimplify sparingly and do not abuse Simplify either.

Furthermore, one often has an aesthetic sense of the form that is desired and there is no guarantee that **Simplify** shares your notion of simplicity or aesthetics. You can then attempt to guide the simplification process by applying transformation functions, or explicit replacement rules, in the order you believe will lead to the desired form. Such a calculation is generally developed by accretion, adding transformations sequentially in postfix form until the goal is reached. Sometimes the same transformation has to be applied several times at different stages of the calculation, perhaps interspersed with simplification commands, with intermediate stages organized using parentheses. Unfortunately, the appropriate order for these steps can be difficult to work out and sometimes changes from one release of *Mathematica* to the next. There are few general rules we can impart — simplification guided by aesthetics is a fine art acquired only by experience. On the other hand, unless you have a good reason to insist upon the simplest possible expressions, simplification is often superfluous. One can calculate and plot functions or investigate many of their properties without getting bogged down in unnecessary manipulations.

Applications on Simplify[expr], Expand[expr], Factor[expr] Apart[expr], Together[expr]

Mathematica can work with expressions as well as numerical input. You can factor, combine like terms, and expand expressions.

In[60]:=

$$4x^2 - 3x + 7 - 8x^2 + 6x^3 + 11x - 9$$

Out[60]=

$$-2 + 8x - 4x^2 + 6x^3$$

In[61]:=

```
Simplify[(7 x^2 + 3 x - 8) - (6 x^2 - 5 x - 24)]
```

Out[61]=

$(4+x)^2$

In[62]:=

```
Expand[(2 x^2 - 3 x + 4)^4]
```

Out[62]=

$256-768\,x+1376\,x^2-1584\,x^3+1329\,x^4-792\,x^5+344\,x^6-96\,x^7+16\,x^8$

In[63]:=

```
Factor[2 x^3 + 13 x^2 - 7 x]
```

Out[63]=

$x\,(7+x)\,(-1+2\,x)$

In[64]:=

$\text{Apart}\left[\frac{5\,x+2}{x^2+5\,x+4}\right]$

Out[64]=

$-\frac{1}{1+x}+\frac{6}{4+x}$

In[65]:=

$$\text{Together}\left[\frac{1}{2\,x+3}-\frac{5\,x}{x^2-1}\right]$$

Out[65]=

$$\frac{-1-15\,x-9\,x^2}{(3+2\,x)\left(-1+x^2\right)}$$

A nice utility in *Mathematica* is the % expression, which represents the last output.

In[66]:=

$$\frac{4\,x^2-5}{2\,x^2-5\,x+2}$$

Out[66]=

$$\frac{-5+4\,x^2}{2-5\,x+2\,x^2}$$

In[67]:=

Apart[%]

Out[67]=

$$2+\frac{11}{3\,(-2+x)}+\frac{8}{3\,(-1+2\,x)}$$

In[68]:=

Together[%]

Out[68]=

$$\frac{-5+4\,x^2}{(-2+x)\,(-1+2\,x)}$$

In[69]:=

```
Expand[%]
```

Out[69]=

$$-\frac{5}{(-2+x)(-1+2x)}+\frac{4x^2}{(-2+x)(-1+2x)}$$

In[70]:=

```
ExpandAll[%]
```

Out[70]=

$$-\frac{5}{2-5x+2x^2}+\frac{4x^2}{2-5x+2x^2}$$

In[71]:=

```
Simplify[%]
```

Out[71]=

$$\frac{-5+4x^2}{2-5x+2x^2}$$

Simplify:

a) $3\sin(x)-\sin(3x)$ b) $\cos(\sin^{-1}(x))$ c) $\frac{3\tanh(x)+\tanh^3(x)}{1+3\tanh^2(x)}$

For the expression $2\cosh^3(x)\sinh(x) + 2\cosh(x)\sinh^3(x)$, compare the following transformations:

a) Factor b) Factor[expr,Trig→True] c) TrigFactor
d) TrigReduce e) Simplify f) Simplify[expr,Trig→False]

Express $\ln\left(\frac{x+1}{x-1}\right)$ in terms of hyperbolic trigonometric functions.

[Hint: you might need the substitution $x \to E^{2y}$ and several steps.] I leave it to you to judge which form is simplest — most statistical physics textbooks express this formula in terms of hyperbolic trigonometric functions, but rational expressions are pretty simple also. Often one's preference depends upon context.

Example

The entropy for a system is proportional to the logarithm of the total number of states with the same energy. Suppose that a system is described by a multiplicity function of the form

In[72]:=

```
multiplicity = (n! / (m! (n - m)!))^2;
```

$$\text{multiplicity} = \left(\frac{n!}{m!\,(n-m)!}\right)^2;$$

where *n* and *m* are very large positive integers, of order 10^{23}, with $m \ll n$. Under those circumstances it is useful to apply the Stirling approximation for Log[*n*!] in the form

In[73]:=

```
StirlingApprox = {Log[x_!] → x Log[x] - x};
```

First, note that **Simplify** is ineffective because it does not recognize that the variables in this expression are positive and that the logarithm can be reduced without specifying the appropriate branch, but **PowerExpand** does help.

In[74]:=

```
Log[multiplicity] // Simplify
```

Out[74]=

$$\text{Log}\left[\frac{(n!)^2}{(m!)^2\,((-m+n)!)^2}\right]$$

In[75]:=

```
Log[multiplicity] // PowerExpand
```

Out[75]=

```
-2 Log[m!] + 2 Log[n!] - 2 Log[(-m + n)!]
```

Next we apply the Stirling approximation. Note that parentheses are included to ensure that **PowerExpand** is executed first. (Try to do without the parentheses.)

In[76]:=

```
(Log[multiplicity] // PowerExpand) /. StirlingApprox
```

Out[76]=

```
-2 (-m + m Log[m]) + 2 (-n + n Log[n]) -
  2 (m - n + (-m + n) Log[-m + n])
```

Because we know that $m < n$, it is convenient to make the substitution $m \to x\,n$ where $0 \leq x \leq 1$ is the ratio between m and n. (The physics would also be discussed in terms of x if this were a course in statistical physics.) Therefore, we append this replacement rule and another simplification step.

In[77]:=

```
(Log[multiplicity] // PowerExpand) /. StirlingApprox /.
    m → x n // Simplify
```

Out[77]=

```
2 n (Log[n] - x Log[n x] + (-1 + x) Log[n - n x])
```

Finally, although we probably could use built-in functions, it is easier to simplify these logarithms with explicit rules, using cut-and-paste editing. It is again necessary to use parentheses to ensure proper order of operations.

In[78]:=

```
((Log[multiplicity] // PowerExpand) /. StirlingApprox /.
        m → x n // Simplify) /. {Log[n x] → Log[n] + Log[x],
    Log[n - n x] → Log[n] + Log[1 - x]} // Simplify
```

Out[78]=

```
2 n ((-1 + x) Log[1 - x] - x Log[x])
```

We have now obtained a tidy expression, and could transform it in other ways if desired. Rather than retaining each step of such a calculation, usually only the last pair of input/output cells would be kept in a notebook. Although the fairly complicated input cell above would then appear to have sprung from the author's forehead fully armed in Athenian splendor, it would actually have been assembled like an oyster's shell one layer at a time.

FunctionExpand

Expressions involving special functions with complicated arguments can often be simplified using FunctionExpand.

In[79]:=

$$\text{FunctionExpand}\left[\text{Sinh}\left[\frac{\text{ArcCosh}[x]}{2}\right]\right]$$

Out[79]=

$$\frac{\sqrt{-1+x}}{\sqrt{2}}$$

In[80]:=

```
FunctionExpand[Cos[4 ArcTan[x]]] // Simplify
```

Out[80]=

$$\frac{1-6\,x^2+x^4}{\left(1+x^2\right)^2}$$

Simplify: $\text{Tan}\left[\frac{\text{ArcCos}[x]}{2}\right]$

Simplification with assumptions

A powerful new feature of simplification became available in *Version 4.0*. It is often possible to simplify symbolic expressions much further when the properties of the various symbols, such as their numerical ranges, are known in advance. The syntax **Simplify[expression,assumptions]** permits a list of **assumptions** to be employed during the simplification of **expression**. The assumptions are specified either as a list, **{assumption1,assumption2,…}** or as a logical expression, such as **assumption1&&assumption2&&assumption3**. Assumptions can specify the type (domain) for various variables, allowed ranges for values, or relationships between variables. Assumptions can be employed with **Simplify**, **FullSimplify**, **FunctionExpand**, **Refine**, **Limit**, or **Integrate.**

Consider the following expression.

In[81]:=

```
expr = Log[a − b] + Log[a + b];
```

In[82]:=

```
Simplify[expr]
```

Out[82]=

```
Log[a − b] + Log[a + b]
```

Simplification is ineffective without further information about *a* and *b*, but if we know that $a > b$

In[83]:=

```
Simplify[expr, {a > b}]
```

Out[83]=

$\text{Log}[a^2 - b^2]$

then the two functions can be combined. Note that the assumption $a > b$ implicitly contains the additional assumption that both a and b are real numbers, so that their magnitudes can be compared directly. Similarly, the following expressions can be simplified by employing inequalities to define ranges.

In[84]:=

$\{\text{Simplify}[\sqrt{(x-y)^2}], \text{Simplify}[\sqrt{(x-y)^2}, x > y]\}$

Out[84]=

$\{\sqrt{(x-y)^2}, x - y\}$

In[85]:=

$\text{Simplify}[2\sqrt{x\,y} \leq x + y, \{x \geq 0, y \geq 0\}]$

Out[85]=

True

In[86]:=

$\text{Limit}[\text{Exp}[x\,y], y \to \infty, \text{Assumptions} \to \{x < 0\}]$

Out[86]=

0

Often it is sufficient to specify a domain (**Integers**, **Rationals**, **Reals**, **Algebraics**, **Complexes**, **Booleans**, **Primes**) for one or more variables. The function Element**[x,domain]** specifies that **x∈ domain** is an element of the specified domain. The element operator ∈ can be entered with the keystroke sequence ESCelemESC. The variations **{x,y,z}∈domain** or **(x|y|z)∈domain** assign each member of a list to the specified domain. Domains can also be specified for patterns according to **patt∈domain**.

In[87]:=

```
{Simplify[Cos[n π]], Simplify[Cos[n π], {n ∈ Integers}]}
```

Out[87]=

$\{\text{Cos}[n\,\pi], (-1)^n\}$

In[88]:=

$\text{Simplify}\left[\text{Cos}[n\,\pi], \left\{\frac{n+1}{2} \in \text{Integers}\right\}\right]$

Out[88]=

-1

A related function is Refine, which uses assumptions to produce output that more closely approximates the form that you might use when its symbols satisfy explicit numerical assumptions. The result is often better than produced by **Simplify**, or at least more specific.

In[89]:=

```
{Simplify[Abs[x], x < 0], Refine[Abs[x], x < 0]}
```

Out[89]=

```
{Abs[x], -x}
```

In[90]:=

```
{Simplify[Log[x], x < 0], Refine[Log[x], x < 0]}
```

Out[90]=

$\{\text{Log}[x], i\,\pi + \text{Log}[-x]\}$

In[91]:=

```
{Simplify[Cos[x + π y], y ∈ Integers],
  Refine[Cos[x + π y], y ∈ Integers]}
```

Out[91]=

$\{\text{Cos}[x + \pi y], (-1)^y \text{Cos}[x]\}$

If one is performing a long derivation involving many symbols, it can become cumbersome to specify a long list of assumptions many times. The following example illustrates a useful technique for performing such calculations. First we create a list a assumptions that apply to the variables of interest. Then we define our own versions of the simplification functions using these assumptions. Two examples using these functions follow.

In[92]:=

```
MyAssumptions =
    {0 < θe < π, ω > 0, q > Q, Q > 0, 0 < ϵ < 1, εf > 0, εi > εf};
MySimplify = Simplify[#, MyAssumptions] &;
MyFullSimplify = FullSimplify[#, MyAssumptions] &;
```

In[95]:=

$$\text{sol1} = \text{Solve}\left[\epsilon == \left(1 + 2\,\frac{q^2}{Q^2}\,\text{Tan}\left[\frac{\theta_e}{2}\right]^2\right)^{-1},\ \theta_e\right][[2]]\ //\ \text{MySimplify}$$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[95]=

$$\left\{\theta_e \to -2\,\text{ArcTan}\left[\frac{Q\sqrt{-1 + \frac{1}{\epsilon}}}{\sqrt{2}\,q}\right]\right\}$$

In[96]:=

```
Solve[Q^2 == 4 εi (εi - ω) Sin[θe/2]^2, εi][[2]] /. ω → Sqrt[q^2 - Q^2] /. sol1 //
  MyFullSimplify
```

Out[96]=

$$\left\{\varepsilon_i \to \frac{1}{2}\left(\sqrt{q^2 - Q^2} + \frac{q\,(1+\epsilon)}{\sqrt{1-\epsilon^2}}\right)\right\}$$

Although the formatting procedure used by *Mathematica* still has some peculiarities (*e.g.*, too many minus signs), these results were simplified fairly completely with relatively little effort, whereas much more complicated expressions would have been produced had we not told *Mathematica* what assumptions it could employ (Try it!). In a real problem we would probably use our simplification routines many more times, perhaps adding additional assumptions as the solution is developed.

Under what assumptions will $(x^m)^n$ reduce to $x^{m\,n}$? Verify. List a few examples which show that uncritical use of the proposed replacement rule leads to incorrect results.

Prove $a^n - b^n = (a-b)\left(a^{n-1} + a^{n-2}\,b + \cdots + b^{n-1}\right)$ for any positive integer n and real a, b.

Symbolic solution of equations

Basic syntax

The basic tools for symbolic solution of algebraic equations are **`Solve`**, **`Reduce`**, and **`Eliminate`**. Solve**`[eqs,vars]`** attempts to solve an equation or list of equations, **`eqs`**, for the variable or list of variables, **vars**. Equations must be expressed in the form **lhs == rhs.**

In[97]:=

```
eq1 = a x^2 + b x + c == 0;
```

In[98]:=

```
sol1 = Solve[eq1, x]
```

Out[98]=

$$\left\{\left\{x \to \frac{-b - \sqrt{b^2 - 4 a c}}{2 a}\right\}, \left\{x \to \frac{-b + \sqrt{b^2 - 4 a c}}{2 a}\right\}\right\}$$

Solutions are returned as sets of replacement rules. In this case there are two possible solutions, each containing a single replacement rule for the only independent variable. A particular solution can then be selected by using the replacement rule of your choice to make an assignment

In[99]:=

```
x1 = x /. sol1[[1]]
x2 = x /. sol1[[2]]
```

Out[99]=

$$\frac{-b - \sqrt{b^2 - 4 a c}}{2 a}$$

Out[100]=

$$\frac{-b + \sqrt{b^2 - 4 a c}}{2 a}$$

Caution: if you use assignments of the form **x=x/.solution** you will contaminate the symbol **x** and may have difficulty revising your equation later.

Sometimes it is useful to verify that solutions are correct by substitution into the original equations. It is usually necessary to perform some simplification steps before the veracity of the equations becomes manifest.

In[101]:=

```
eq1 /. sol1 // Simplify
```

Out[101]=

```
{True, True}
```

A system of equations is expressed in the form of a list of equations, {*lhs1* == *rhs1*, *lhs2* == *rhs2*, ⋯} or as an expression in which equations are combined using **And** (**&&**) in the form *lhs1*== *rhs1* && *lhs2*== *rhs2* ⋯.

In[102]:=

$$\text{Solve}\left[\left\{3\,x^2 - 2\,y^2 == 1,\; x^2 + 4\,y^2 == 3\right\}, \{x, y\}\right]$$

Out[102]=

$$\left\{\left\{x \to -\sqrt{\frac{5}{7}},\, y \to -\frac{2}{\sqrt{7}}\right\}, \left\{x \to -\sqrt{\frac{5}{7}},\, y \to \frac{2}{\sqrt{7}}\right\}, \left\{x \to \sqrt{\frac{5}{7}},\, y \to -\frac{2}{\sqrt{7}}\right\}, \left\{x \to \sqrt{\frac{5}{7}},\, y \to \frac{2}{\sqrt{7}}\right\}\right\}$$

Now **Solve** returns four solutions, each consisting of two replacement rules. In other cases, degenerate solutions are duplicated.

In[103]:=

$$\text{Solve}\left[36 + 12\,x - 11\,x^2 - 2\,x^3 + x^4 == 0,\, x\right]$$

Out[103]=

$$\{\{x \to -2\}, \{x \to -2\}, \{x \to 3\}, \{x \to 3\}\}$$

Solve does not check for special cases that might arise for some choices of parameters. For example, the solution proposed for the following equation

In[104]:=

```
Solve[a x^2 + b x == 1, x]
```

Out[104]=

$$\left\{\left\{x \to \frac{-b - \sqrt{4 a + b^2}}{2 a}\right\}, \left\{x \to \frac{-b + \sqrt{4 a + b^2}}{2 a}\right\}\right\}$$

does not accommodate the special case $a \to 0$. More complete solutions are provided by Reduce, which checks special cases and returns solutions in the form of logical conditions upon both variables and parameters.

In[105]:=

```
Reduce[a x^2 + b x == 1, x]
```

Out[105]=

$$\left(a \neq 0 \;\&\&\; \left(x == \frac{-b - \sqrt{4 a + b^2}}{2 a} \;||\; x == \frac{-b + \sqrt{4 a + b^2}}{2 a}\right)\right) \;||$$
$$\left(a == 0 \;\&\&\; b \neq 0 \;\&\&\; x == \frac{1}{b}\right)$$

Thus, we recover the original solutions when $a \neq 0$, but in addition find a solution for the special case of $a = 0$ also. (Actually, the solution for $a \to 0$ can be obtained by Taylor expansion of the general case but **Reduce** requires less work.)

Eliminate eliminates a variable or set of variables from a system of equations and returns a smaller system combined with logical operators. Often this is useful when it is convenient to formulate a system of equations in terms of one or more intermediate quantities that help clarify the relationships between the independent variables. Sometimes it is useful for transformation of variables. For example, the following expression transforms a hyperbola and a line from Cartesian to polar coordinates.

In[106]:=

```
Eliminate[
  {x^2 - y^2 == a^2, x + y == b, x == r Cos[θ], y == r Sin[θ]}, {x, y}]
```

Out[106]=

$b == r\,(\mathrm{Cos}[\theta] + \mathrm{Sin}[\theta])\ \&\&\ r^2\,\mathrm{Cos}[\theta]^2 == a^2 + r^2\,\mathrm{Sin}[\theta]^2$

Solve[eqs,vars,elims] first eliminates variables in **elims** and then solves the resulting equations for **vars**, thereby combining the functionality of **Solve** and **Eliminate** in a single expression.

Solve the preceding system of equations for {r,θ} and simplify assuming that {a,b,r,θ} are all positive.

Find the roots of LegendreP[6,x] symbolically and verify that these roots are, in fact, real.

A parabola is the locus of all points in a plane that are equidistant from a fixed point, called the *focus*, and a fixed line, called the *directrix*. Construct the parabola whose focus is given by *focus={a,b}* and whose directrix is the *x*-axis. Then determine the coordinates of the *vertex*, which is the point closest to the directrix.

High-order polynomial equations

Solve produces symbolic or exact numerical solutions to polynomial equations of degree 4 or less, but will also return symbolic roots to higher-order polynomials as well. Suppose that we wish to determine the fixed points for the 4th iteration of the logistic map, which are defined by the following equation.

Nest[*f*, *expr*, *n*] gives an expression with *f* applied *n* times to *expr*.

In[107]:=

```
eq1 = x == (Nest[μ # (1 - #) &, x, 4])
```

Out[107]=

$$x == (1-x)\,x\,\mu^4\,(1-(1-x)\,x\,\mu)\,\left(1-(1-x)\,x\,\mu^2\,(1-(1-x)\,x\,\mu)\right)\left(1-(1-x)\,x\,\mu^3\,(1-(1-x)\,x\,\mu)\left(1-(1-x)\,x\,\mu^2\,(1-(1-x)\,x\,\mu)\right)\right)$$

In[108]:=

```
eq2 = x == (Nest[μ # (1 - #) &, x, 4] // Expand)
```

Out[108]=

$$x == x\,\mu^4 - x^2\,\mu^4 - x^2\,\mu^5 + 2\,x^3\,\mu^5 - x^4\,\mu^5 - x^2\,\mu^6 + 2\,x^3\,\mu^6 - x^4\,\mu^6 - x^2\,\mu^7 + 4\,x^3\,\mu^7 - 7\,x^4\,\mu^7 + 6\,x^5\,\mu^7 - 2\,x^6\,\mu^7 + 2\,x^3\,\mu^8 - 7\,x^4\,\mu^8 + 10\,x^5\,\mu^8 - 8\,x^6\,\mu^8 + 4\,x^7\,\mu^8 - x^8\,\mu^8 + 2\,x^3\,\mu^9 - 7\,x^4\,\mu^9 + 10\,x^5\,\mu^9 - 8\,x^6\,\mu^9 + 4\,x^7\,\mu^9 - x^8\,\mu^9 - 6\,x^4\,\mu^{10} + 24\,x^5\,\mu^{10} - 36\,x^6\,\mu^{10} + 24\,x^7\,\mu^{10} - 6\,x^8\,\mu^{10} - x^4\,\mu^{11} + 10\,x^5\,\mu^{11} - 36\,x^6\,\mu^{11} + 64\,x^7\,\mu^{11} - 61\,x^8\,\mu^{11} + 30\,x^9\,\mu^{11} - 6\,x^{10}\,\mu^{11} + 4\,x^5\,\mu^{12} - 22\,x^6\,\mu^{12} + 52\,x^7\,\mu^{12} - 70\,x^8\,\mu^{12} + 60\,x^9\,\mu^{12} - 34\,x^{10}\,\mu^{12} + 12\,x^{11}\,\mu^{12} - 2\,x^{12}\,\mu^{12} - 6\,x^6\,\mu^{13} + 36\,x^7\,\mu^{13} - 90\,x^8\,\mu^{13} + 120\,x^9\,\mu^{13} - 90\,x^{10}\,\mu^{13} + 36\,x^{11}\,\mu^{13} - 6\,x^{12}\,\mu^{13} + 4\,x^7\,\mu^{14} - 28\,x^8\,\mu^{14} + 84\,x^9\,\mu^{14} - 140\,x^{10}\,\mu^{14} + 140\,x^{11}\,\mu^{14} - 84\,x^{12}\,\mu^{14} + 28\,x^{13}\,\mu^{14} - 4\,x^{14}\,\mu^{14} - x^8\,\mu^{15} + 8\,x^9\,\mu^{15} - 28\,x^{10}\,\mu^{15} + 56\,x^{11}\,\mu^{15} - 70\,x^{12}\,\mu^{15} + 56\,x^{13}\,\mu^{15} - 28\,x^{14}\,\mu^{15} + 8\,x^{15}\,\mu^{15} - x^{16}\,\mu^{15}$$

Although there exist no general methods for solving equations of order 16, **Solve** will nonetheless return Root objects which can be evaluated numerically. (Open to view)

In[109]:=

```
fixedpoints = Solve[eq2, x]
```

Out[109]=

$$\left\{\{x \to 0\}, \left\{x \to \frac{-1+\mu}{\mu}\right\}, \left\{x \to \frac{\mu+\mu^2-\mu\sqrt{-3-2\mu+\mu^2}}{2\mu^2}\right\},\right.$$

$$\left\{x \to \frac{\mu+\mu^2+\mu\sqrt{-3-2\mu+\mu^2}}{2\mu^2}\right\},$$

$$\{x \to \text{Root}[1+\mu^2+\mu^2(-1-\mu-\mu^2-\mu^3)\#1+ \mu^3(2+\mu+4\mu^2+\mu^3+2\mu^4)\#1^2+ \mu^3(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6)\#1^3+ \mu^5(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5)\#1^4+ \mu^6(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5)\#1^5+ \mu^6(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6)\#1^6+ \mu^8(-2-14\mu-12\mu^2-30\mu^3-6\mu^4)\#1^7+ \mu^9(6+3\mu+30\mu^2+15\mu^3)\#1^8+ \mu^9(-1-15\mu^2-20\mu^3)\#1^9+ \mu^{11}(3+15\mu)\#1^{10}-6\mu^{12}\#1^{11}+\mu^{12}\#1^{12}\ \&,\ 1]\},$$

$$\{x \to \text{Root}[1+\mu^2+\mu^2(-1-\mu-\mu^2-\mu^3)\#1+ \mu^3(2+\mu+4\mu^2+\mu^3+2\mu^4)\#1^2+ \mu^3(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6)\#1^3+ \mu^5(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5)\#1^4+ \mu^6(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5)\#1^5+ \mu^6(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6)\#1^6+ \mu^8(-2-14\mu-12\mu^2-30\mu^3-6\mu^4)\#1^7+ \mu^9(6+3\mu+30\mu^2+15\mu^3)\#1^8+ \mu^9(-1-15\mu^2-20\mu^3)\#1^9+ \mu^{11}(3+15\mu)\#1^{10}-6\mu^{12}\#1^{11}+\mu^{12}\#1^{12}\ \&,\ 2]\},$$

$$\{x \to \text{Root}[1+\mu^2+\mu^2(-1-\mu-\mu^2-\mu^3)\#1+ \mu^3(2+\mu+4\mu^2+\mu^3+2\mu^4)\#1^2+$$

$$\mu^3\left(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6\right)\sharp1^3+$$
$$\mu^5\left(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5\right)\sharp1^4+$$
$$\mu^6\left(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5\right)\sharp1^5+$$
$$\mu^6\left(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6\right)\sharp1^6+$$
$$\mu^8\left(-2-14\mu-12\mu^2-30\mu^3-6\mu^4\right)\sharp1^7+$$
$$\mu^9\left(6+3\mu+30\mu^2+15\mu^3\right)\sharp1^8+$$
$$\mu^9\left(-1-15\mu^2-20\mu^3\right)\sharp1^9+$$
$$\mu^{11}(3+15\mu)\sharp1^{10}-6\mu^{12}\sharp1^{11}+\mu^{12}\sharp1^{12}\ \&,\ 3\Big]\Big\},$$

$$\Big\{x\to\text{Root}\Big[1+\mu^2+\mu^2\left(-1-\mu-\mu^2-\mu^3\right)\sharp1+$$
$$\mu^3\left(2+\mu+4\mu^2+\mu^3+2\mu^4\right)\sharp1^2+$$
$$\mu^3\left(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6\right)\sharp1^3+$$
$$\mu^5\left(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5\right)\sharp1^4+$$
$$\mu^6\left(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5\right)\sharp1^5+$$
$$\mu^6\left(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6\right)\sharp1^6+$$
$$\mu^8\left(-2-14\mu-12\mu^2-30\mu^3-6\mu^4\right)\sharp1^7+$$
$$\mu^9\left(6+3\mu+30\mu^2+15\mu^3\right)\sharp1^8+$$
$$\mu^9\left(-1-15\mu^2-20\mu^3\right)\sharp1^9+$$
$$\mu^{11}(3+15\mu)\sharp1^{10}-6\mu^{12}\sharp1^{11}+\mu^{12}\sharp1^{12}\ \&,\ 4\Big]\Big\},$$

$$\Big\{x\to\text{Root}\Big[1+\mu^2+\mu^2\left(-1-\mu-\mu^2-\mu^3\right)\sharp1+$$
$$\mu^3\left(2+\mu+4\mu^2+\mu^3+2\mu^4\right)\sharp1^2+$$
$$\mu^3\left(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6\right)\sharp1^3+$$
$$\mu^5\left(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5\right)\sharp1^4+$$
$$\mu^6\left(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5\right)\sharp1^5+$$
$$\mu^6\left(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6\right)\sharp1^6+$$
$$\mu^8\left(-2-14\mu-12\mu^2-30\mu^3-6\mu^4\right)\sharp1^7+$$
$$\mu^9\left(6+3\mu+30\mu^2+15\mu^3\right)\sharp1^8+$$
$$\mu^9\left(-1-15\mu^2-20\mu^3\right)\sharp1^9+$$

$\mu^{11}(3+15\mu)\,\#1^{10}-6\,\mu^{12}\,\#1^{11}+\mu^{12}\,\#1^{12}\,\&,\ 5]\},$

$\{x\to \text{Root}[1+\mu^2+\mu^2(-1-\mu-\mu^2-\mu^3)\,\#1+$
$\mu^3(2+\mu+4\mu^2+\mu^3+2\mu^4)\,\#1^2+$
$\mu^3(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6)\,\#1^3+$
$\mu^5(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5)\,\#1^4+$
$\mu^6(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5)\,\#1^5+$
$\mu^6(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6)\,\#1^6+$
$\mu^8(-2-14\mu-12\mu^2-30\mu^3-6\mu^4)\,\#1^7+$
$\mu^9(6+3\mu+30\mu^2+15\mu^3)\,\#1^8+$
$\mu^9(-1-15\mu^2-20\mu^3)\,\#1^9+$
$\mu^{11}(3+15\mu)\,\#1^{10}-6\,\mu^{12}\,\#1^{11}+\mu^{12}\,\#1^{12}\,\&,\ 6]\},$

$\{x\to \text{Root}[1+\mu^2+\mu^2(-1-\mu-\mu^2-\mu^3)\,\#1+$
$\mu^3(2+\mu+4\mu^2+\mu^3+2\mu^4)\,\#1^2+$
$\mu^3(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6)\,\#1^3+$
$\mu^5(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5)\,\#1^4+$
$\mu^6(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5)\,\#1^5+$
$\mu^6(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6)\,\#1^6+$
$\mu^8(-2-14\mu-12\mu^2-30\mu^3-6\mu^4)\,\#1^7+$
$\mu^9(6+3\mu+30\mu^2+15\mu^3)\,\#1^8+$
$\mu^9(-1-15\mu^2-20\mu^3)\,\#1^9+$
$\mu^{11}(3+15\mu)\,\#1^{10}-6\,\mu^{12}\,\#1^{11}+\mu^{12}\,\#1^{12}\,\&,\ 7]\},$

$\{x\to \text{Root}[1+\mu^2+\mu^2(-1-\mu-\mu^2-\mu^3)\,\#1+$
$\mu^3(2+\mu+4\mu^2+\mu^3+2\mu^4)\,\#1^2+$
$\mu^3(-1-5\mu^2-4\mu^3-5\mu^4-4\mu^5-\mu^6)\,\#1^3+$
$\mu^5(2+6\mu+4\mu^2+14\mu^3+5\mu^4+3\mu^5)\,\#1^4+$
$\mu^6(-4-\mu-18\mu^2-12\mu^3-12\mu^4-3\mu^5)\,\#1^5+$
$\mu^6(1+10\mu^2+17\mu^3+18\mu^4+15\mu^5+\mu^6)\,\#1^6+$

$$\mu^8 \left(-2 - 14\,\mu - 12\,\mu^2 - 30\,\mu^3 - 6\,\mu^4\right) \#1^7 +$$
$$\mu^9 \left(6 + 3\,\mu + 30\,\mu^2 + 15\,\mu^3\right) \#1^8 +$$
$$\mu^9 \left(-1 - 15\,\mu^2 - 20\,\mu^3\right) \#1^9 +$$
$$\mu^{11} (3 + 15\,\mu) \#1^{10} - 6\,\mu^{12} \#1^{11} + \mu^{12} \#1^{12} \,\&,\, 8\big]\big\},$$
$$\big\{x \to \text{Root}\big[1 + \mu^2 + \mu^2 \left(-1 - \mu - \mu^2 - \mu^3\right) \#1 +$$
$$\mu^3 \left(2 + \mu + 4\,\mu^2 + \mu^3 + 2\,\mu^4\right) \#1^2 +$$
$$\mu^3 \left(-1 - 5\,\mu^2 - 4\,\mu^3 - 5\,\mu^4 - 4\,\mu^5 - \mu^6\right) \#1^3 +$$
$$\mu^5 \left(2 + 6\,\mu + 4\,\mu^2 + 14\,\mu^3 + 5\,\mu^4 + 3\,\mu^5\right) \#1^4 +$$
$$\mu^6 \left(-4 - \mu - 18\,\mu^2 - 12\,\mu^3 - 12\,\mu^4 - 3\,\mu^5\right) \#1^5 +$$
$$\mu^6 \left(1 + 10\,\mu^2 + 17\,\mu^3 + 18\,\mu^4 + 15\,\mu^5 + \mu^6\right) \#1^6 +$$
$$\mu^8 \left(-2 - 14\,\mu - 12\,\mu^2 - 30\,\mu^3 - 6\,\mu^4\right) \#1^7 +$$
$$\mu^9 \left(6 + 3\,\mu + 30\,\mu^2 + 15\,\mu^3\right) \#1^8 +$$
$$\mu^9 \left(-1 - 15\,\mu^2 - 20\,\mu^3\right) \#1^9 +$$
$$\mu^{11} (3 + 15\,\mu) \#1^{10} - 6\,\mu^{12} \#1^{11} + \mu^{12} \#1^{12} \,\&,\, 9\big]\big\},$$
$$\big\{x \to \text{Root}\big[1 + \mu^2 + \mu^2 \left(-1 - \mu - \mu^2 - \mu^3\right) \#1 +$$
$$\mu^3 \left(2 + \mu + 4\,\mu^2 + \mu^3 + 2\,\mu^4\right) \#1^2 +$$
$$\mu^3 \left(-1 - 5\,\mu^2 - 4\,\mu^3 - 5\,\mu^4 - 4\,\mu^5 - \mu^6\right) \#1^3 +$$
$$\mu^5 \left(2 + 6\,\mu + 4\,\mu^2 + 14\,\mu^3 + 5\,\mu^4 + 3\,\mu^5\right) \#1^4 +$$
$$\mu^6 \left(-4 - \mu - 18\,\mu^2 - 12\,\mu^3 - 12\,\mu^4 - 3\,\mu^5\right) \#1^5 +$$
$$\mu^6 \left(1 + 10\,\mu^2 + 17\,\mu^3 + 18\,\mu^4 + 15\,\mu^5 + \mu^6\right) \#1^6 +$$
$$\mu^8 \left(-2 - 14\,\mu - 12\,\mu^2 - 30\,\mu^3 - 6\,\mu^4\right) \#1^7 +$$
$$\mu^9 \left(6 + 3\,\mu + 30\,\mu^2 + 15\,\mu^3\right) \#1^8 +$$
$$\mu^9 \left(-1 - 15\,\mu^2 - 20\,\mu^3\right) \#1^9 + \mu^{11} (3 + 15\,\mu) \#1^{10} -$$
$$6\,\mu^{12} \#1^{11} + \mu^{12} \#1^{12} \,\&,\, 10\big]\big\},$$
$$\big\{x \to \text{Root}\big[1 + \mu^2 + \mu^2 \left(-1 - \mu - \mu^2 - \mu^3\right) \#1 +$$
$$\mu^3 \left(2 + \mu + 4\,\mu^2 + \mu^3 + 2\,\mu^4\right) \#1^2 +$$
$$\mu^3 \left(-1 - 5\,\mu^2 - 4\,\mu^3 - 5\,\mu^4 - 4\,\mu^5 - \mu^6\right) \#1^3 +$$

$\mu^5 (2 + 6\mu + 4\mu^2 + 14\mu^3 + 5\mu^4 + 3\mu^5)\,\#1^4 +$
$\mu^6 (-4 - \mu - 18\mu^2 - 12\mu^3 - 12\mu^4 - 3\mu^5)\,\#1^5 +$
$\mu^6 (1 + 10\mu^2 + 17\mu^3 + 18\mu^4 + 15\mu^5 + \mu^6)\,\#1^6 +$
$\mu^8 (-2 - 14\mu - 12\mu^2 - 30\mu^3 - 6\mu^4)\,\#1^7 +$
$\mu^9 (6 + 3\mu + 30\mu^2 + 15\mu^3)\,\#1^8 +$
$\mu^9 (-1 - 15\mu^2 - 20\mu^3)\,\#1^9 + \mu^{11} (3 + 15\mu)\,\#1^{10} -$
$6\mu^{12}\,\#1^{11} + \mu^{12}\,\#1^{12}\ \&, 11]\},$

$\{x \to \text{Root}[1 + \mu^2 + \mu^2 (-1 - \mu - \mu^2 - \mu^3)\,\#1 +$
$\mu^3 (2 + \mu + 4\mu^2 + \mu^3 + 2\mu^4)\,\#1^2 +$
$\mu^3 (-1 - 5\mu^2 - 4\mu^3 - 5\mu^4 - 4\mu^5 - \mu^6)\,\#1^3 +$
$\mu^5 (2 + 6\mu + 4\mu^2 + 14\mu^3 + 5\mu^4 + 3\mu^5)\,\#1^4 +$
$\mu^6 (-4 - \mu - 18\mu^2 - 12\mu^3 - 12\mu^4 - 3\mu^5)\,\#1^5 +$
$\mu^6 (1 + 10\mu^2 + 17\mu^3 + 18\mu^4 + 15\mu^5 + \mu^6)\,\#1^6 +$
$\mu^8 (-2 - 14\mu - 12\mu^2 - 30\mu^3 - 6\mu^4)\,\#1^7 +$
$\mu^9 (6 + 3\mu + 30\mu^2 + 15\mu^3)\,\#1^8 +$
$\mu^9 (-1 - 15\mu^2 - 20\mu^3)\,\#1^9 + \mu^{11} (3 + 15\mu)\,\#1^{10} -$
$6\mu^{12}\,\#1^{11} + \mu^{12}\,\#1^{12}\ \&, 12]\}\}$

If you want to know the number of elements in fixedpoints, you use

`Length[expr]` gives the number of elements in *expr*.

In[110]:=

```
Length[fixedpoints]
```

Out[110]=

```
16
```

The list begins with roots that can be obtained by factoring and is followed by **Root** objects representing the roots of the remaining irreducible polynomial. Each **Root** object has two arguments: a pure function represents the irreducible part of the polynomial and an index identifies a particular root. These roots can be displayed for a range of μ using **Plot**.

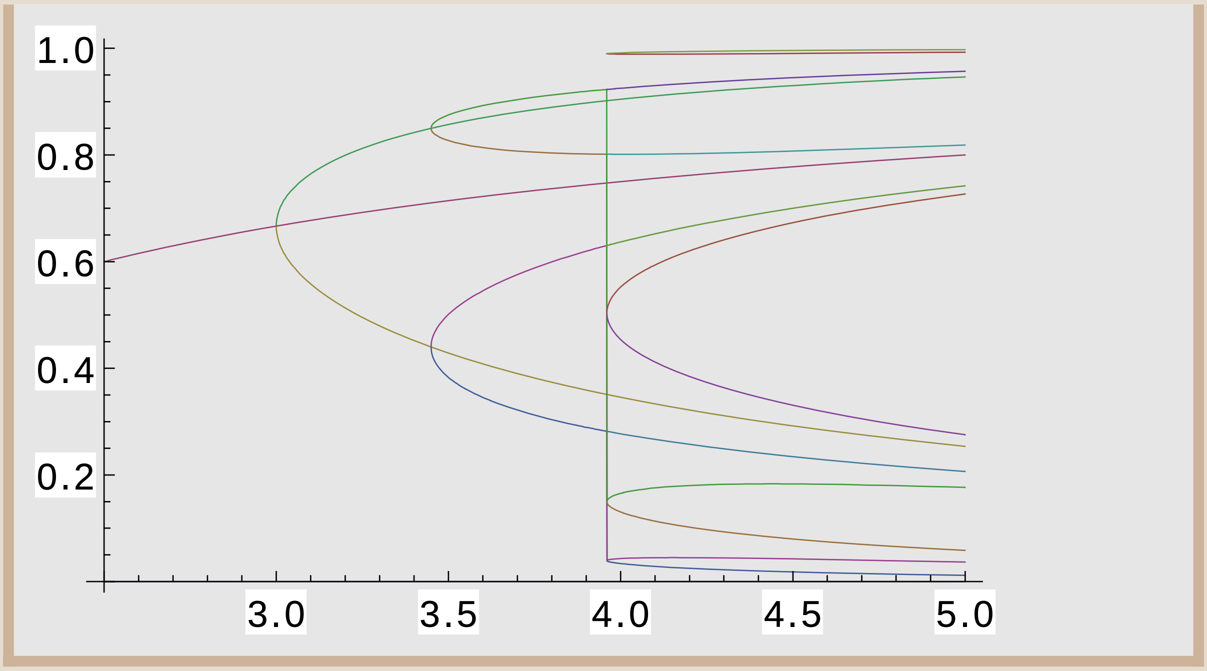
NumberMarks

is an option for InputForm and related functions that specifies whether ` marks should be included in the printed forms of approximate numbers.

In[111]:=

```
Plot[Evaluate[x /. fixedpoints], {μ, 2.5`, 5}]
```

Out[111]=



The error messages are generated for roots that are complex for some range of μ and are no cause for alarm. As μ increases more of these roots become real and gradually assemble the pitchfork bifurcation pattern for the logistic map. More detail about the nature of period doubling and the onset of chaos within deterministic systems can be found in ***cobweb.nb***.

Solve the following equation for the quantity r=y/x and plot the real solutions for {a,-1,1}:

$$\frac{2}{y^4} == \frac{1}{x^4} + \frac{a}{(x+y)^4}$$

[Hint: substitute y→r x first and then simplify the equation as much as possible. (You may need to use Map to perform the same transformations to both sides of the equation.)]

Algebraic equations in transcendental disguise

Solve is designed to handle polynomial equations, but will often attempt to find solutions for more general equations. For example, in the trivial example

In[112]:=

```
Solve[Cos[x] == Sin[x], x]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[112]=

$$\left\{\left\{x \to -\frac{3\pi}{4}\right\}, \left\{x \to \frac{\pi}{4}\right\}\right\}$$

two solutions are reported, but the 2π periodicity is overlooked when only the principal branches of inverse trigonometric functions are used. However, in other cases inverse functions do not any provide solutions at all and numerical solutions must be sought (see below).

In[113]:=

```
Solve[x == Tan[x], x]
```

Solve::tdep :
The equations appear to involve the variables to be solved for in an essentially non–algebraic way. ≫

Out[113]=

```
Solve[x == Tan[x], x]
```

Examples

Example: y-scaling for quasielastic scattering

Suppose that a high-energy electron scatters from one of the nucleons in a nucleus with mass m_A, delivering energy transfer ω and momentum transfer q, such that a nucleon of mass m_N is ejected leaving a residual nucleus of mass m_B. In the spectator model, the target nucleus is represented at the time of the interaction as the struck nucleon with initial momentum y plus spectator m_B with momentum $-y$, such that the ejectile emerges with final momentum $y+q$. Conservation of energy (relativistic) then requires

In[114]:=

$$\text{eq1} = \omega + m_A == \sqrt{m_N^2 + (y+q)^2} + \sqrt{m_B^2 + y^2};$$

Don't worry if you are unfamiliar with the physics behind this equation — we only pose it as an example of a nontrivial algebraic equation which we would rather solve using *Mathematica* than by hand.

In[115]:=

```
sol1 = Solve[eq1, y] // Simplify
```

Out[115]=

$$\left\{\left\{y \to -\frac{1}{2\left(-q^2+\omega^2+2\,\omega\, m_A+m_A^2\right)}\left(-q^3+q\,\omega^2+2\,q\,\omega\, m_A+q\, m_A^2+q\, m_B^2-q\, m_N^2+\sqrt{\left((\omega+m_A)^2\left(4\,\omega\, m_A^3+m_A^4+m_B^4+2\, m_B^2\left(q^2-\omega^2-m_N^2\right)+\left(q^2-\omega^2+m_N^2\right)^2-2\, m_A^2\left(q^2-3\,\omega^2+m_B^2+m_N^2\right)-4\,\omega\, m_A\left(q^2-\omega^2+m_B^2+m_N^2\right)\right)\right)}\right)\right\},\right.$$

$$\left.\left\{y \to -\frac{1}{2\left(-q^2+\omega^2+2\,\omega\, m_A+m_A^2\right)}\left(-q^3+q\,\omega^2+2\,q\,\omega\, m_A+q\, m_A^2+q\, m_B^2-q\, m_N^2-\sqrt{\left((\omega+m_A)^2\left(4\,\omega\, m_A^3+m_A^4+m_B^4+2\, m_B^2\left(q^2-\omega^2-m_N^2\right)+\left(q^2-\omega^2+m_N^2\right)^2-2\, m_A^2\left(q^2-3\,\omega^2+m_B^2+m_N^2\right)-4\,\omega\, m_A\left(q^2-\omega^2+m_B^2+m_N^2\right)\right)\right)}\right)\right\}\right\}$$

Mathematica produces two solutions, but the ordering of these solutions is to some degree arbitrary and varies from version to version. We must select the appropriate solution using additional infor mation, such as a physical interpretation or constraint. For this problem, it is useful to specialize to a target which consists only of a single nucleon without any residual spectator. Furthermore, we know that the momentum transfer must be greater than or equal to the energy transfer. Hence, it is useful to solve for ω in terms of q for this special case.

We use **Thread** to produce new equations for this special case from the list of possible solutions for the general case.

In[116]:=

```
newEqs = Thread[0 == y /. sol1 /. {m_A → m_N, m_B → 0}]
```

Out[116]=

$$\left\{0 == -\left(-q^3 + q\,\omega^2 + 2\,q\,\omega\,m_N + \sqrt{\left((\omega + m_N)^2 \left(4\,\omega\,m_N^3 + m_N^4 - 2\,m_N^2\left(q^2 - 3\,\omega^2 + m_N^2\right) - 4\,\omega\,m_N\left(q^2 - \omega^2 + m_N^2\right) + \left(q^2 - \omega^2 + m_N^2\right)^2\right)\right)}\right) \Big/ \left(2\left(-q^2 + \omega^2 + 2\,\omega\,m_N + m_N^2\right)\right),\ 0 == -\left(-q^3 + q\,\omega^2 + 2\,q\,\omega\,m_N - \sqrt{\left((\omega + m_N)^2 \left(4\,\omega\,m_N^3 + m_N^4 - 2\,m_N^2\left(q^2 - 3\,\omega^2 + m_N^2\right) - 4\,\omega\,m_N\left(q^2 - \omega^2 + m_N^2\right) + \left(q^2 - \omega^2 + m_N^2\right)^2\right)\right)}\right) \Big/ \left(2\left(-q^2 + \omega^2 + 2\,\omega\,m_N + m_N^2\right)\right)\right\}$$

Nested expansion and simplification steps are needed to print attractive versions of these equations, but that does not affect their solution.

In[117]:=

```
newEqs = (newEqs // Simplify // PowerExpand) // Simplify
```

Out[117]=

$$\left\{\frac{q^2 - \omega^2 - 2\,\omega\,m_N}{q - \omega - m_N} == 0,\ \frac{-q^2 + \omega^2 + 2\,\omega\,m_N}{q + \omega + m_N} == 0\right\}$$

In[118]:=

```
Solve[newEqs, ω]
```

Out[118]=

$$\left\{\left\{\omega \to -m_N - \sqrt{q^2 + m_N^2}\right\}, \left\{\omega \to -m_N + \sqrt{q^2 + m_N^2}\right\}\right\}$$

It is now clear that the second solution is the one that we want.

Example: nonrelativistic binary collisions

A nonrelativistic binary collision is described by a mass-balance equation of the form $m1 + m2 == m3 + m4$, where we consider *m1* to be the projectile, *m2* the target, *m3* the scattered particle, and *m4* the scattered target, but it is also possible that the reactants exchange mass during the collision. The kinematics of the reaction are governed by conservation of momentum and of energy, which are expressed by the following equations.

In[119]:=

$$\text{collisionEqs} = \left\{m1 + m2 == m3 + m4,\ p1 + p2 == p3 + p4,\ \frac{p1.p1}{2\,m1} + \frac{p2.p2}{2\,m2} + \text{qvalue} == \frac{p3.p3}{2\,m3} + \frac{p4.p4}{2\,m4}\right\};$$

Here the *qvalue* measures the inelasticity of the reaction and vanishes for elastic scattering. Note that we use Dot, with operator notation ., to indicate the scalar product of two vectors. The momenta are defined using a set of replacement rules. Note that the since the reaction is confined to a plane, we can omit the z-components and work with vectors in two dimensions.

In[120]:=

```
momentumComponentRules =
    {p1 → {p1x, p1y}, p2 → {p2x, p2y}, p3 → {p3x, p3y},
      p4 → {p4x, p4y}, p1x → p1r Cos[θ1], p1y → p1r Sin[θ1],
      p2x → p2r Cos[θ2], p2y → p2r Sin[θ2],
      p3x → p3r Cos[θ3], p3y → p3r Sin[θ3]};
```

Because there are 4 independent equations, we can solve this system for 4 variables. It is convenient to specify three masses *{m1,m2,m3}*, the projectile momentum *{p1x,p1y}* in terms of magnitude *p1r* and angle *theta1*, the target momentum *{p2x,p2y}* in terms of magnitude *p2r* and angle *theta2*, and the projectile scattering angle *theta3*. *Mathematica* will then happily report a general solution for the remaining mass *m4*, the momentum *p3r* for the scattered particle, and the momentum *{p4x,p4y}* of the recoil particle. Although we could use **Simplify**, the minimal return is not worth the considerable investment of time required to simplify expressions of this complexity!

In[121]:=

```
gensol = Solve[collisionEqs //. momentumComponentRules,
    {m4, p3r, p4x, p4y}]
```

Out[121]=

$$\{\{p4x \to p1r\,\mathrm{Cos}[\theta 1] + p2r\,\mathrm{Cos}[\theta 2] - \left(m1\,m2\,m3\,p1r\,\mathrm{Cos}[\theta 1]\,\mathrm{Cos}[\theta 3]^2\right) \Big/ \left(m1^2\,m2\,\mathrm{Cos}[\theta 3]^2 + m1\,m2^2\,\mathrm{Cos}[\theta 3]^2 + m1^2\,m2\,\mathrm{Sin}[\theta 3]^2 + m1\,m2^2\,\mathrm{Sin}[\theta 3]^2\right) - \left(m1\,m2\,m3\,p2r\,\mathrm{Cos}[\theta 2]\,\mathrm{Cos}[\theta 3]^2\right) \Big/ \left(m1^2\,m2\,\mathrm{Cos}[\theta 3]^2 + m1\,m2^2\,\mathrm{Cos}[\theta 3]^2 + m1^2\,m2\,\mathrm{Sin}[\theta 3]^2 + m1\,m2^2\,\mathrm{Sin}[\theta 3]^2\right) - \left(m1\,m2\,m3\,p1r\,\mathrm{Cos}[\theta 3]\,\mathrm{Sin}[\theta 1]\,\mathrm{Sin}[\theta 3]\right) \Big/ \left(m1^2\,m2\,\mathrm{Cos}[\theta 3]^2 + m1\,m2^2\,\mathrm{Cos}[\theta 3]^2 +\right.$$

```
          m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2) -
      (m1 m2 m3 p2r Cos[θ3] Sin[θ2] Sin[θ3]) /
       (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
          m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2) -
      (Cos[θ3] √((-2 m1 m2 m3 p1r Cos[θ1] Cos[θ3] -
                  2 m1 m2 m3 p2r Cos[θ2] Cos[θ3] -
                  2 m1 m2 m3 p1r Sin[θ1] Sin[θ3] -
                  2 m1 m2 m3 p2r Sin[θ2] Sin[θ3])^2 -
               4 (-2 m1^2 m2 m3 qvalue - 2 m1 m2^2 m3
                    qvalue + 2 m1 m2 m3^2 qvalue -
                  m2^2 m3 p1r^2 Cos[θ1]^2 + m2 m3^2
                    p1r^2 Cos[θ1]^2 + 2 m1 m2 m3 p1r
                    p2r Cos[θ1] Cos[θ2] - m1^2 m3 p2r^2
                    Cos[θ2]^2 + m1 m3^2 p2r^2 Cos[θ2]^2 -
                  m2^2 m3 p1r^2 Sin[θ1]^2 + m2 m3^2
                    p1r^2 Sin[θ1]^2 + 2 m1 m2 m3 p1r
                    p2r Sin[θ1] Sin[θ2] - m1^2 m3 p2r^2
                    Sin[θ2]^2 + m1 m3^2 p2r^2 Sin[θ2]^2)
                (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
                  m1^2 m2 Sin[θ3]^2 +
                  m1 m2^2 Sin[θ3]^2))) /
       (2 (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
              m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2)),
  p4y → p1r Sin[θ1] + p2r Sin[θ2] -
      (m1 m2 m3 p1r Cos[θ1]
          Cos[θ3] Sin[θ3]) /
       (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
          m1^2 m2 Sin[θ3]^2 +
          m1 m2^2 Sin[θ3]^2) -
```

$$(\text{m1 m2 m3 p2r Cos}[\theta 2]\ \text{Cos}[\theta 3]\ \text{Sin}[\theta 3]) \Big/$$
$$\left(\text{m1}^2\ \text{m2 Cos}[\theta 3]^2 + \text{m1 m2}^2\ \text{Cos}[\theta 3]^2 + \right.$$
$$\left. \text{m1}^2\ \text{m2 Sin}[\theta 3]^2 + \text{m1 m2}^2\ \text{Sin}[\theta 3]^2\right) -$$
$$\left(\text{m1 m2 m3 p1r Sin}[\theta 1]\ \text{Sin}[\theta 3]^2\right) \Big/$$
$$\left(\text{m1}^2\ \text{m2 Cos}[\theta 3]^2 + \text{m1 m2}^2\ \text{Cos}[\theta 3]^2 + \right.$$
$$\left. \text{m1}^2\ \text{m2 Sin}[\theta 3]^2 + \text{m1 m2}^2\ \text{Sin}[\theta 3]^2\right) -$$
$$\left(\text{m1 m2 m3 p2r Sin}[\theta 2]\ \text{Sin}[\theta 3]^2\right) \Big/$$
$$\left(\text{m1}^2\ \text{m2 Cos}[\theta 3]^2 + \text{m1 m2}^2\ \text{Cos}[\theta 3]^2 + \right.$$
$$\left. \text{m1}^2\ \text{m2 Sin}[\theta 3]^2 + \text{m1 m2}^2\ \text{Sin}[\theta 3]^2\right) -$$
$$\Big(\text{Sin}[\theta 3]\ \sqrt{\Big((-2\ \text{m1 m2 m3 p1r Cos}[\theta 1]\ \text{Cos}[\theta 3] -}$$
$$2\ \text{m1 m2 m3 p2r Cos}[\theta 2]\ \text{Cos}[\theta 3] -$$
$$2\ \text{m1 m2 m3 p1r Sin}[\theta 1]\ \text{Sin}[\theta 3] -$$
$$2\ \text{m1 m2 m3 p2r Sin}[\theta 2]\ \text{Sin}[\theta 3])^2 -$$
$$4\left(-2\ \text{m1}^2\ \text{m2 m3 qvalue} - 2\ \text{m1 m2}^2\ \text{m3}\right.$$
$$\text{qvalue} + 2\ \text{m1 m2 m3}^2\ \text{qvalue} -$$
$$\text{m2}^2\ \text{m3 p1r}^2\ \text{Cos}[\theta 1]^2 + \text{m2 m3}^2$$
$$\text{p1r}^2\ \text{Cos}[\theta 1]^2 + 2\ \text{m1 m2 m3 p1r}$$
$$\text{p2r Cos}[\theta 1]\ \text{Cos}[\theta 2] - \text{m1}^2\ \text{m3 p2r}^2$$
$$\text{Cos}[\theta 2]^2 + \text{m1 m3}^2\ \text{p2r}^2\ \text{Cos}[\theta 2]^2 -$$
$$\text{m2}^2\ \text{m3 p1r}^2\ \text{Sin}[\theta 1]^2 + \text{m2 m3}^2$$
$$\text{p1r}^2\ \text{Sin}[\theta 1]^2 + 2\ \text{m1 m2 m3 p1r}$$
$$\text{p2r Sin}[\theta 1]\ \text{Sin}[\theta 2] - \text{m1}^2\ \text{m3 p2r}^2$$
$$\left.\text{Sin}[\theta 2]^2 + \text{m1 m3}^2\ \text{p2r}^2\ \text{Sin}[\theta 2]^2\right)$$
$$\left(\text{m1}^2\ \text{m2 Cos}[\theta 3]^2 + \text{m1 m2}^2\ \text{Cos}[\theta 3]^2 + \right.$$
$$\text{m1}^2\ \text{m2 Sin}[\theta 3]^2 +$$
$$\left.\text{m1 m2}^2\ \text{Sin}[\theta 3]^2\right)\Big)}\Big) \Big/$$
$$\left(2\left(\text{m1}^2\ \text{m2 Cos}[\theta 3]^2 + \text{m1 m2}^2\ \text{Cos}[\theta 3]^2 + \right.\right.$$
$$\left.\left.\text{m1}^2\ \text{m2 Sin}[\theta 3]^2 + \text{m1 m2}^2\ \text{Sin}[\theta 3]^2\right)\right),$$
$$\text{p3r} \to \left(2\ \text{m1 m2 m3 p1r Cos}[\theta 1]\ \text{Cos}[\theta 3] + \right.$$

```
2 m1 m2 m3 p2r
   Cos[θ2] Cos[θ3] +
2 m1 m2 m3 p1r Sin[θ1] Sin[θ3] +
2 m1 m2 m3 p2r
   Sin[θ2] Sin[θ3] +
√((-2 m1 m2 m3 p1r Cos[θ1] Cos[θ3] -
      2 m1 m2 m3 p2r Cos[θ2] Cos[θ3] -
      2 m1 m2 m3 p1r Sin[θ1] Sin[θ3] -
      2 m1 m2 m3 p2r Sin[θ2] Sin[θ3])^2 -
   4 (-2 m1^2 m2 m3 qvalue - 2 m1 m2^2
         m3 qvalue + 2 m1 m2 m3^2 qvalue -
      m2^2 m3 p1r^2 Cos[θ1]^2 +
      m2 m3^2 p1r^2 Cos[θ1]^2 +
      2 m1 m2 m3 p1r p2r Cos[θ1] Cos[θ2] -
      m1^2 m3 p2r^2 Cos[θ2]^2 +
      m1 m3^2 p2r^2 Cos[θ2]^2 - m2^2 m3 p1r^2
         Sin[θ1]^2 + m2 m3^2 p1r^2 Sin[θ1]^2 +
      2 m1 m2 m3 p1r p2r Sin[θ1] Sin[θ2] -
      m1^2 m3 p2r^2 Sin[θ2]^2 +
      m1 m3^2 p2r^2 Sin[θ2]^2)
    (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
      m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2))) /
(2 (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
      m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2)),
m4 → m1 + m2 - m3},
{p4x →
   p1r
      Cos[
        θ1] + p2r Cos[
        θ2] -
```

$$
\begin{aligned}
&(m1\ m2\ m3\ p1r\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 3]^2) / \\
&\quad (m1^2\ m2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1^2\ m2\ \mathrm{Sin}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Sin}[\theta 3]^2) - \\
&(m1\ m2\ m3\ p2r\ \mathrm{Cos}[\theta 2]\ \mathrm{Cos}[\theta 3]^2) / \\
&\quad (m1^2\ m2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1^2\ m2\ \mathrm{Sin}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Sin}[\theta 3]^2) - \\
&(m1\ m2\ m3\ p1r\ \mathrm{Cos}[\theta 3]\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 3]) / \\
&\quad (m1^2\ m2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1^2\ m2\ \mathrm{Sin}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Sin}[\theta 3]^2) - \\
&(m1\ m2\ m3\ p2r\ \mathrm{Cos}[\theta 3]\ \mathrm{Sin}[\theta 2]\ \mathrm{Sin}[\theta 3]) / \\
&\quad (m1^2\ m2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Cos}[\theta 3]^2 + \\
&\qquad m1^2\ m2\ \mathrm{Sin}[\theta 3]^2 + \\
&\qquad m1\ m2^2\ \mathrm{Sin}[\theta 3]^2) + \\
&(\mathrm{Cos}[\theta 3]\ \sqrt{}((-2\ m1\ m2\ m3\ p1r\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 3] - \\
&\qquad\qquad 2\ m1\ m2\ m3\ p2r\ \mathrm{Cos}[\theta 2]\ \mathrm{Cos}[\theta 3] - \\
&\qquad\qquad 2\ m1\ m2\ m3\ p1r\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 3] - \\
&\qquad\qquad 2\ m1\ m2\ m3\ p2r\ \mathrm{Sin}[\theta 2]\ \mathrm{Sin}[\theta 3])^2 - \\
&\qquad 4\,(-2\ m1^2\ m2\ m3\ qvalue - 2\ m1\ m2^2\ m3 \\
&\qquad\qquad qvalue + 2\ m1\ m2\ m3^2\ qvalue - \\
&\qquad\qquad m2^2\ m3\ p1r^2\ \mathrm{Cos}[\theta 1]^2 + m2\ m3^2 \\
&\qquad\qquad p1r^2\ \mathrm{Cos}[\theta 1]^2 + 2\ m1\ m2\ m3\ p1r \\
&\qquad\qquad p2r\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 2] - m1^2\ m3\ p2r^2
\end{aligned}
$$

```
Cos[θ2]^2 + m1 m3^2 p2r^2 Cos[θ2]^2 -
m2^2 m3 p1r^2 Sin[θ1]^2 + m2 m3^2
p1r^2 Sin[θ1]^2 + 2 m1 m2 m3 p1r
p2r Sin[θ1] Sin[θ2] - m1^2 m3 p2r^2
Sin[θ2]^2 + m1 m3^2 p2r^2 Sin[θ2]^2)
(m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
m1^2 m2 Sin[θ3]^2 +
m1 m2^2 Sin[θ3]^2)))/
(2 (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2)),
p4y → p1r Sin[θ1] + p2r Sin[θ2] -
(m1 m2 m3 p1r Cos[θ1]
Cos[θ3] Sin[θ3])/
(m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
m1^2 m2 Sin[θ3]^2 +
m1 m2^2 Sin[θ3]^2) -
(m1 m2 m3 p2r Cos[θ2] Cos[θ3] Sin[θ3])/
(m1^2 m2 Cos[θ3]^2 +
m1 m2^2 Cos[θ3]^2 +
m1^2 m2 Sin[θ3]^2 +
m1 m2^2 Sin[θ3]^2) -
(m1 m2 m3 p1r Sin[θ1] Sin[θ3]^2)/
(m1^2 m2 Cos[θ3]^2 +
m1 m2^2 Cos[θ3]^2 +
m1^2 m2 Sin[θ3]^2 +
m1 m2^2 Sin[θ3]^2) -
(m1 m2 m3 p2r Sin[θ2] Sin[θ3]^2)/
(m1^2 m2 Cos[θ3]^2 +
m1 m2^2 Cos[θ3]^2 +
```

$$
\begin{aligned}
&\mathrm{m1}^2\ \mathrm{m2}\ \mathrm{Sin}[\theta 3]^2 + \\
&\mathrm{m1}\ \mathrm{m2}^2\ \mathrm{Sin}[\theta 3]^2\Big) + \\
&\Big(\mathrm{Sin}[\theta 3]\ \sqrt{\Big(}(-2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r}\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 3] - \\
&\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p2r}\ \mathrm{Cos}[\theta 2]\ \mathrm{Cos}[\theta 3] - \\
&\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r}\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 3] - \\
&\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p2r}\ \mathrm{Sin}[\theta 2]\ \mathrm{Sin}[\theta 3])^2 - \\
&\quad 4\Big(-2\ \mathrm{m1}^2\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{qvalue} - 2\ \mathrm{m1}\ \mathrm{m2}^2\ \mathrm{m3} \\
&\quad\quad \mathrm{qvalue} + 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}^2\ \mathrm{qvalue} - \\
&\quad \mathrm{m2}^2\ \mathrm{m3}\ \mathrm{p1r}^2\ \mathrm{Cos}[\theta 1]^2 + \mathrm{m2}\ \mathrm{m3}^2 \\
&\quad\quad \mathrm{p1r}^2\ \mathrm{Cos}[\theta 1]^2 + 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r} \\
&\quad\quad \mathrm{p2r}\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 2] - \mathrm{m1}^2\ \mathrm{m3}\ \mathrm{p2r}^2 \\
&\quad\quad \mathrm{Cos}[\theta 2]^2 + \mathrm{m1}\ \mathrm{m3}^2\ \mathrm{p2r}^2\ \mathrm{Cos}[\theta 2]^2 - \\
&\quad \mathrm{m2}^2\ \mathrm{m3}\ \mathrm{p1r}^2\ \mathrm{Sin}[\theta 1]^2 + \mathrm{m2}\ \mathrm{m3}^2 \\
&\quad\quad \mathrm{p1r}^2\ \mathrm{Sin}[\theta 1]^2 + 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r} \\
&\quad\quad \mathrm{p2r}\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 2] - \mathrm{m1}^2\ \mathrm{m3}\ \mathrm{p2r}^2 \\
&\quad\quad \mathrm{Sin}[\theta 2]^2 + \mathrm{m1}\ \mathrm{m3}^2\ \mathrm{p2r}^2\ \mathrm{Sin}[\theta 2]^2\Big) \\
&\quad \Big(\mathrm{m1}^2\ \mathrm{m2}\ \mathrm{Cos}[\theta 3]^2 + \mathrm{m1}\ \mathrm{m2}^2\ \mathrm{Cos}[\theta 3]^2 + \\
&\quad \mathrm{m1}^2\ \mathrm{m2}\ \mathrm{Sin}[\theta 3]^2 + \\
&\quad \mathrm{m1}\ \mathrm{m2}^2\ \mathrm{Sin}[\theta 3]^2\Big)\Big)\Big) \Big/ \\
&\Big(2\Big(\mathrm{m1}^2\ \mathrm{m2}\ \mathrm{Cos}[\theta 3]^2 + \mathrm{m1}\ \mathrm{m2}^2\ \mathrm{Cos}[\theta 3]^2 + \\
&\quad \mathrm{m1}^2\ \mathrm{m2}\ \mathrm{Sin}[\theta 3]^2 + \mathrm{m1}\ \mathrm{m2}^2\ \mathrm{Sin}[\theta 3]^2\Big)\Big), \\
&\mathrm{p3r} \to \Big(2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r}\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 3] + \\
&\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p2r} \\
&\quad\quad \mathrm{Cos}[\theta 2]\ \mathrm{Cos}[\theta 3] + \\
&\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r}\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 3] + \\
&\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p2r} \\
&\quad\quad \mathrm{Sin}[\theta 2]\ \mathrm{Sin}[\theta 3] - \\
&\quad \sqrt{\Big(}(-2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r}\ \mathrm{Cos}[\theta 1]\ \mathrm{Cos}[\theta 3] - \\
&\quad\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p2r}\ \mathrm{Cos}[\theta 2]\ \mathrm{Cos}[\theta 3] - \\
&\quad\quad 2\ \mathrm{m1}\ \mathrm{m2}\ \mathrm{m3}\ \mathrm{p1r}\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 3] -
\end{aligned}
$$

```
                2 m1 m2 m3 p2r Sin[θ2] Sin[θ3])^2 -
            4 (-2 m1^2 m2 m3 qvalue - 2 m1 m2^2
                m3 qvalue + 2 m1 m2 m3^2 qvalue -
                m2^2 m3 p1r^2 Cos[θ1]^2 +
                m2 m3^2 p1r^2 Cos[θ1]^2 +
                2 m1 m2 m3 p1r p2r Cos[θ1] Cos[θ2] -
                m1^2 m3 p2r^2 Cos[θ2]^2 +
                m1 m3^2 p2r^2 Cos[θ2]^2 - m2^2 m3 p1r^2
                Sin[θ1]^2 + m2 m3^2 p1r^2 Sin[θ1]^2 +
                2 m1 m2 m3 p1r p2r Sin[θ1] Sin[θ2] -
                m1^2 m3 p2r^2 Sin[θ2]^2 +
                m1 m3^2 p2r^2 Sin[θ2]^2)
            (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
                m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2))) /
      (2 (m1^2 m2 Cos[θ3]^2 + m1 m2^2 Cos[θ3]^2 +
            m1^2 m2 Sin[θ3]^2 + m1 m2^2 Sin[θ3]^2)),
  m4 → m1 + m2 - m3}}
```

Familiar special cases can be examined by choosing appropriate parameters. For example, consider the elastic scattering of two particles of equal mass assuming that the target is at rest.

In[122]:=

```
(gensol /. {m1 → m, m2 → m, m3 → m, qvalue → 0, theta1 → 0,
      p2r → 0} // Simplify // PowerExpand) // Simplify
```

Out[122]=

```
{{p4x → -p1r Sin[θ1 - θ3] Sin[θ3],
    p4y → p1r Cos[θ3] Sin[θ1 - θ3],
    p3r → p1r Cos[θ1 - θ3], m4 → m},
  {p4x → p1r Cos[θ1], p4y → p1r Sin[θ1], p3r → 0, m4 → m}}
```

Clearly the first solution is the one we seek while the second solution is degenerate, describing only the special case of a head-on collision in which the projectile transfers all of its momentum to the struck particle. More general situations can now be investigated by varying the parameters in the first solution.

Numerical solution of equations

Polynomial equations

An equation or system of equations involving only polynomials can be solved numerically using NSolve**[eqs,vars,n]** where **eqs** is an equation or list of equations, **vars** is the variable or list of variables, and **n** is an optional parameter specifying the precision sought in terms of number of decimal digits. The result is returned as a list of sets of replacement rules.

In[123]:=

```
NSolve[x^7 - 3 x^3 == x - 2, x]
```

Out[123]=

```
{{x → -1.42937}, {x → -0.425843 - 0.812252 i},
  {x → -0.425843 + 0.812252 i}, {x → 0.115351 - 1.29244 i},
  {x → 0.115351 + 1.29244 i}, {x → 0.774278}, {x → 1.27608}}
```

In[124]:=

```
NSolve[{x^2 y - x == 1, y^2 + x y == -2}, {x, y}]
```

Out[124]=

$\{\{x \to 0.40198 - 0.835793\, i,\ y \to -0.25844 + 1.87993\, i\},$
$\{x \to 0.40198 + 0.835793\, i,\ y \to -0.25844 - 1.87993\, i\},$
$\{x \to -0.568647 + 0.253329\, i,\ y \to 0.25844 + 1.26469\, i\},$
$\{x \to -0.568647 - 0.253329\, i,\ y \to 0.25844 - 1.26469\, i\}\}$

NSolve will also handle equations which are really polynomials in disguise

In[125]:=

$$\text{NSolve}\left[\left\{\sqrt{x^2 y - y^2 x} == \sqrt[4]{1 + x y},\ x\sqrt{x - y} == 2\right\}, \{x, y\}\right]$$

Out[125]=

$\{\{x \to 3.69376,\ y \to 3.40059\},\ \{x \to 1.86251,\ y \to 0.709423\}\}$

and many others for which inverse functions are available

In[126]:=

```
NSolve[Cos[x]^2 - Sin[x]^2 == 0.5, x]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[126]=

$\{\{x \to -2.61799\},\ \{x \to -0.523599\},\ \{x \to 0.523599\},\ \{x \to 2.61799\}\}$

In[127]:=

$$\text{NSolve}\left[e^{2x} - e^{x} == 1, x\right]$$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

Out[127]=

$$\{\{x \to -0.481212 + 3.14159\, i\}, \{x \to 0.481212\}\}$$

The warning message about inverse functions is usually nothing to worry about, and can be disabled using **Off**, but after solving a problem one should go back and determine whether alternative solutions have been overlooked.

However, **NSolve** is unable to solve equations which involve transcendental functions in an essentially nonalgebraic manner that makes transformation to polynomial form impossible.

In[128]:=

$$\text{lhs} = \frac{\xi}{\sqrt{1-\xi^2}};\quad \text{rhs} = \text{Cot}[\xi];$$

$$\text{eq1} = \text{lhs} == \text{rhs};$$

In[130]:=

$$\text{NSolve}[\text{eq1}, \xi]$$

Solve::tdep :
The equations appear to involve the variables to be solved for in an essentially non–algebraic way. ≫

Out[130]=

$$\text{NSolve}\left[\frac{\xi}{\sqrt{1-\xi^2}} == \text{Cot}[\xi], \xi\right]$$

Write a function using NSolve and Cases which returns the real solutions to the following equation in terms of the parameter *a* :

$$2 == r^4 + \frac{a\, r^4}{(1+r)^4}$$

Plot these solutions for {a,-1,1}. [Hint: it is probably easiest to use ListPlot for this purpose.]

Transcendental equations

The more general function FindRoot can be used to find numerical solutions to equations or systems of equations given appropriate starting conditions. For a single equation, **FindRoot[eq,** $\{x, x_0\}$**]** uses a variant of the secant method to search for a solution in the vicinity of x_0 while **FindRoot[*eq*,** $\{x, \{x_0, x_1\}\}$**]** uses Newton's method based upon two starting values $\{x_0, x_1\}$. The secant method can be used when symbolic derivatives are possible, but Newton's method must be used otherwise. Also note that **FindRoot[expr,** $\{x, x_0\}$**]** where *expr* is an expression rather than an equation is equivalent to **FindRoot[expr==0,** $\{x, x_0\}$**]**.

For the simple equation below we recognize by inspection that there is only one root for $y > 0$ and, hence, need not be too careful in selecting the starting value for the secant method.

In[131]:=

```
eq1 = y Tanh[y] == 1;
```

In[132]:=

```
sol1 = FindRoot[eq1, {y, 0.1}]
```

Out[132]=

```
{y → 1.19968}
```

To test the accuracy of the solution, we change the equation to a difference between its two sides and then substitute the solution.

In[133]:=

```
eq1 /. Equal → Subtract /. sol1
```

Out[133]=

-1.11022×10^{-16}

Even though our starting value is poor, the accuracy of the solution is quite good; note that the solution contains more significant figures than are routinely printed.

For more complicated equations with several roots, we should compare the left-hand and right-hand sides graphically to choose the appropriate starting conditions.

In[134]:=

$$\text{lhs} = \frac{\xi}{\sqrt{16-\xi^2}}$$

$$\text{rhs} = \text{Cot}[\xi]$$

$$\text{eq1} = \text{lhs} == \text{rhs}$$

Out[134]=

$$\frac{\xi}{\sqrt{16-\xi^2}}$$

Out[135]=

$$\text{Cot}[\xi]$$

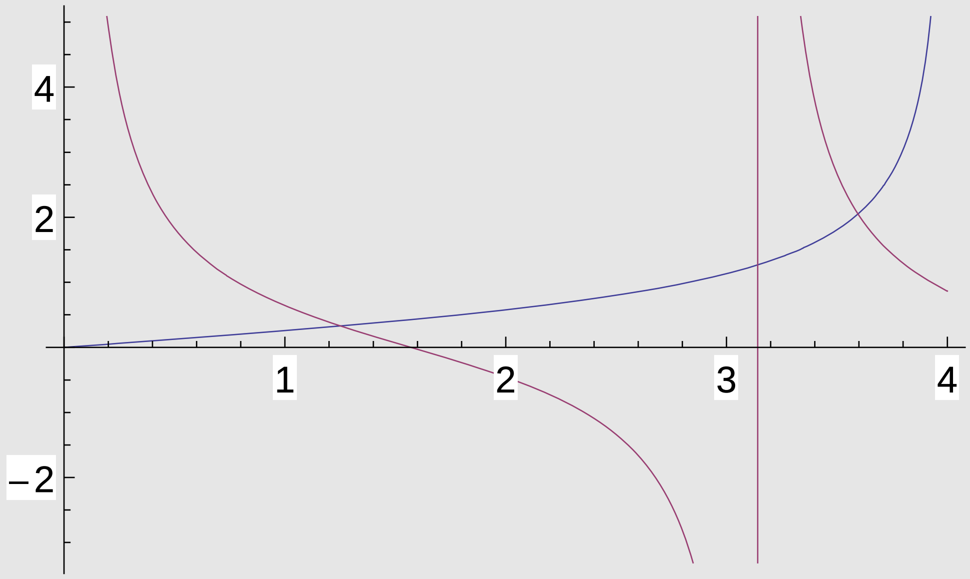
Out[136]=

$$\frac{\xi}{\sqrt{16-\xi^2}} == \text{Cot}[\xi]$$

In[137]:=

```
Plot[Evaluate[{lhs, rhs}], {ξ, 0, 4}]
```

Out[137]=



Here we find two positive roots by supplying two sets of starting conditions based upon the figure.

In[138]:=

```
FindRoot[eq1, {ξ, #}] & /@ {1, 3.5}
```

Out[138]=

```
{{ξ → 1.25235}, {ξ → 3.5953}}
```

Find the first 3 positive roots of $\tan(x) = 1/x$.

Find the first 3 positive roots of BesselJ[2,x]. Then explore the sensitivity to starting values; for example, compare results for starting points of 6.8, 7.0, 7.2.

Find all real solutions to the pair of equations $x^4 + y^4 = 1$ and $e^x - e^y = 1$. [Hint: use ImplicitPlot to display the two equations and to locate appropriate starting values.]

Linear algebra

Matrix manipulation

A matrix is represented as a list of sublists in which each sublist has the same size. *Mathematica* provides many built-in functions with obvious names for manipulating and analyzing matrices. The most important matrix functions are summarized in the accompanying table.

function	output
Array[*mat*, {*m*, *n*}]	$m \times n$ matrix with elements *mat*[*i*, *j*]
DiagonalMatrix[*list*]	diagonal matrix using elements from *list*
IdentityMatrix[*m*]	*m* – dimensional unit matrix
MatrixForm	print matrix in traditional form
Transpose	interchange rows and columns
Inverse[*A*]	inverse matrix A^{-1}
Dot[*A*, *B*]	product *A.B* for compatible matrices
Det	determinant of square matrix
Eigenvalues[*A*]	list of eigenvalues λ, such that A.x = λ x
Eigenvectors[*A*]	list of eigenvectors x, such that A.x = λ x
Eigensystem	list containing both list of eigenvalues and list of eigenvectors
LinearSolve[*A*, *rhs*]	solution vector *x* to A.x == rhs
NullSpace[*A*]	list of vectors *x* for which A.x == 0
MatrixPower[*A*, *b*]	A^b = matrix *A* raised to power *b*
MatrixExp[*A*]	Exp[*A*] as defined by power series

A few examples of some of these functions are given below.

In[139]:=

```
matrix = Array[m, {3, 3}]
```

Out[139]=

```
{{m[1, 1], m[1, 2], m[1, 3]},
  {m[2, 1], m[2, 2], m[2, 3]}, {m[3, 1], m[3, 2], m[3, 3]}}
```

In[140]:=

```
matrix // MatrixForm
```

Out[140]//MatrixForm=

$$\begin{pmatrix} m[1,1] & m[1,2] & m[1,3] \\ m[2,1] & m[2,2] & m[2,3] \\ m[3,1] & m[3,2] & m[3,3] \end{pmatrix}$$

In[141]:=

```
matrix // Transpose // MatrixForm
```

Out[141]//MatrixForm=

$$\begin{pmatrix} m[1,1] & m[2,1] & m[3,1] \\ m[1,2] & m[2,2] & m[3,2] \\ m[1,3] & m[2,3] & m[3,3] \end{pmatrix}$$

The product of two matrices with compatible dimensions is formed by Dot with operator form (.).

In[142]:=

```
var = Array[v, {3}];
```

In[143]:=

```
matrix.var
```

Out[143]=

```
{m[1, 1] v[1] + m[1, 2] v[2] + m[1, 3] v[3],
  m[2, 1] v[1] + m[2, 2] v[2] + m[2, 3] v[3],
  m[3, 1] v[1] + m[3, 2] v[2] + m[3, 3] v[3]}
```

Evaluate MatrixExp[{{0,θ},{-θ,0}}] and interpret the result.

Matrix form of linear equations

A system of equations of the form $A.x == b$ can be formed using **Thread** to move the function **Equal** through the head **List**. (What happens if **Thread** is omitted?)

In[144]:=

```
matrix = Array[A, {3, 3}];
var = Array[x, {3}];
rhs = Array[b, {3}];
```

In[147]:=

```
Thread[matrix.var == rhs]
```

Out[147]=

```
{A[1, 1] x[1] + A[1, 2] x[2] + A[1, 3] x[3] == b[1],
  A[2, 1] x[1] + A[2, 2] x[2] + A[2, 3] x[3] == b[2],
  A[3, 1] x[1] + A[3, 2] x[2] + A[3, 3] x[3] == b[3]}
```

The result is then a form suitable for **Solve**. Alternatively, **LogicalExpand**

In[148]:=

```
LogicalExpand[matrix.var == rhs]
```

Out[148]=

```
A[1, 1] x[1] + A[1, 2] x[2] + A[1, 3] x[3] == b[1] &&
  A[2, 1] x[1] + A[2, 2] x[2] + A[2, 3] x[3] == b[2] &&
  A[3, 1] x[1] + A[3, 2] x[2] + A[3, 3] x[3] == b[3]
```

also produces an expression suitable for input to **Solve**. Although these methods are most often used for systems of linear equations, they can be employed for more general problems if the coefficient matrix and variable vector are constructed properly.

Alternatively, LinearSolve**[coeff,rhs]** is a specialized version for linear equations of the form **coeff.x==rhs**. The solution is a vector **x** which is returned as a simple list, rather than as a set of replacement rules; note that it is not necessary to supply variable names. **LinearSolve** works with either symbolic or numerical expressions. For underdetermined systems **LinearSolve** returns only one of the possible solutions, whereas **Solve** returns the general solution. Also note that for sparse matrices it is usually more efficient to use **Solve**.

In[149]:=

$$\text{coeff} = \left\{\left\{1, -2, \frac{3}{8}\right\}, \{2, 2, -3\}, \left\{\frac{1}{2}, -1, 4\right\}\right\};$$

```
rhs = {-1, 3, 0};
LinearSolve[coeff, rhs]
```

Out[151]=

$$\left\{\frac{143}{183}, \frac{335}{366}, \frac{8}{61}\right\}$$

The null space of a matrix is defined by a linear combination of basis vectors satisfying **matrix.m==0** and is obtained using Null-

Space**[matrix]**.

In[152]:=

```
a = {{1, 1, 0}, {-1, 0, 0}, {1, 1, 0}};
b = NullSpace[a]
```

Out[153]=

```
{{0, 0, 1}}
```

In[154]:=

```
a.b[[1]]
```

Out[154]=

```
{0, 0, 0}
```

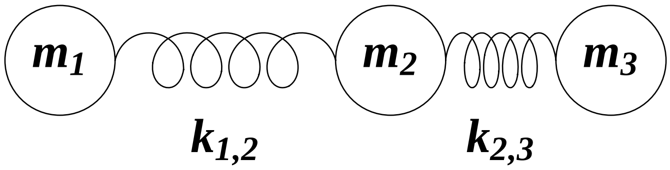
Note that **LinearSolve** and **NullSpace** can be used for rectangular as well as square matrices, as illustrated by the following exercise.

Let a={{1,2,1},{-1,-1,2}} and b={2,3}. Determine vectors *x* and *c* such that *a.(x+λ c)==b* for any value of λ. [Hint: use both LinearSolve and NullSpace.]

Eigensystems

The determinant of a square matrix is given by Det. Eigenvalues finds the eigenvalues and Eigenvectors the eigenvectors of a matrix, either symbolically or numerically. Repeated eigenvalues appear with the requisite multiplicity. Eigensystem returns both in the form of a list with two sublists, the first being the eigenvalues and the second the corresponding eigenvectors. Eigenvectors are not normalized automatically. The package **LinearAlgebra`Orthogonalization`** provides tools to create orthonormal bases. Rather than attempt to describe all available functions in general terms we prefer to use an example.

Example: linear triatomic molecule



Suppose that three masses move along a line subject to pairwise forces represented by springs. The figure above displays the interactions between neighboring masses, but omits a possible interaction between the outer masses with spring constant $k_{1,3}$. One can assume, without loss of generality, that the resting lengths of both interior springs are equal to b. For simplicity we begin by assuming that the two outer masses are equal, $m_3 = m_1$, but not necessarily equal to the central mass m_2, and that $k_{1,2} = k_{2,3}$ while $k_{1,3} = 0$. With these simplifications we can obtain simple but nontrivial symbolic expressions for the frequencies and normal modes of vibration; two different methods are used to illustrate several *Mathematica* functions. Numerical results can also be obtained easily for more general systems, but symbolic expressions become unwieldy.

In[155]:=

```
ClearAll["Global`*"];
Needs["Notation`"];
```

The equations of motion for the general system, assuming that the spring force is proportional to its length change, are obtained from Newton's second law as follows.

In[157]:=

```
equations = {

    m₁ ẍ₁ == k₁,₂ (x₂ − x₁ − b) + k₁,₃ (x₃ − x₁ − 2 b),
            m₂ ẍ₂ == k₂,₃ (x₃ − x₂ − b) − k₁,₂ (x₂ − x₁ − b),
            m₃ ẍ₃ == k₂,₃ (x₂ − x₃ + b) − k₁,₃ (x₃ − x₁ − 2 b)};
```

For our initial special case, these equations reduce to:

In[158]:=

$$eq1 = equations \,/.\, \{m_3 \to m_1,\ k_{1,2} \to k,\ k_{2,3} \to k,\ k_{1,3} \to 0\}$$

Out[158]=

$$\{m_1\, \ddot{x}_1 == k\,(-b - x_1 + x_2),\ m_2\, \ddot{x}_2 == -k\,(-b - x_1 + x_2) + k\,(-b - x_2 + x_3),\ m_1\, \ddot{x}_3 == k\,(b + x_2 - x_3)\}$$

It is useful to eliminate the resting length with the following change of variables.

In[159]:=

$$eq2 = eq1 \,/.\, \{x_1 \to \eta_1 - b,\ x_2 \to \eta_2,\ x_3 \to \eta_3 + b,\ \ddot{x}_1 \to \ddot{\eta}_1,\ \ddot{x}_2 \to \ddot{\eta}_2,\ \ddot{x}_3 \to \ddot{\eta}_3\}$$

Out[159]=

$$\{m_1\, \ddot{\eta}_1 == k\,(-\eta_1 + \eta_2),\ m_2\, \ddot{\eta}_2 == -k\,(-\eta_1 + \eta_2) + k\,(-\eta_2 + \eta_3),\ m_1\, \ddot{\eta}_3 == k\,(\eta_2 - \eta_3)\}$$

Normal modes are defined to be solution vectors of the form $\eta(t) = \eta(0)\,\mathrm{Exp}[-i\,\omega\,t]$. Thus, if we make the substitution $\ddot{\eta} \to -\omega^2\,\eta$ for each coordinate, the time dependence divides out. It is also convenient to replace the equations by differences so that we can extract the coefficient matrix using **`Coefficient`**.

In[160]:=

$$eq3 = eq2 \,/.\, \{\ddot{\eta}_1 \to -\omega^2\,\eta_1,\ \ddot{\eta}_2 \to -\omega^2\,\eta_2,\ \ddot{\eta}_3 \to -\omega^2\,\eta_3\}$$

Out[160]=

$$\{-\omega^2\, m_1\, \eta_1 == k\,(-\eta_1 + \eta_2),\ -\omega^2\, m_2\, \eta_2 == -k\,(-\eta_1 + \eta_2) + k\,(-\eta_2 + \eta_3),\ -\omega^2\, m_1\, \eta_3 == k\,(\eta_2 - \eta_3)\}$$

In[161]:=

```
matrix =
    Coefficient[#, {η1, η2, η3}] & /@ (eq3 /. Equal → Subtract);
matrix // MatrixForm
```

Out[162]//MatrixForm=

$$\begin{pmatrix} k - \omega^2 m_1 & -k & 0 \\ -k & 2k - \omega^2 m_2 & -k \\ 0 & -k & k - \omega^2 m_1 \end{pmatrix}$$

Therefore, the vibrational frequencies for the normal modes are solutions to the equations **matrix.η==0**, which is equivalent to the condition **Det[matrix]==0**. For an *n*-dimensional system, this condition yields an n^{th} degree polynomial equation known as the *secular* or *characteristic equation* for the system. To avoid negative frequencies in the present problem, it is easiest to solve for ω^2 instead of ω itself and to take the positive root later.

In[163]:=

```
freqRule2 = Solve[Det[matrix /. ω^2 → ω2] == 0, ω2] // Simplify
```

Out[163]=

$$\left\{\{\omega 2 \to 0\}, \left\{\omega 2 \to \frac{k}{m_1}\right\}, \left\{\omega 2 \to k\left(\frac{1}{m_1} + \frac{2}{m_2}\right)\right\}\right\}$$

In[164]:=

```
freqRule = ω → Sqrt[ω2] /. freqRule2
```

Out[164]=

$$\left\{\omega \to 0, \omega \to \sqrt{\frac{k}{m_1}}, \omega \to \sqrt{k\left(\frac{1}{m_1} + \frac{2}{m_2}\right)}\right\}$$

The modes are then found by substituting the frequencies back into the equations of motion and solving for the amplitudes.

Because the overall amplitude is arbitrary for linear systems, it is useful to set one of the amplitudes to unity (but not the central one because of the symmetry of the problem).

In[165]:=

```
Table[freqRule[[i]] /. η1 → 1, {i, 1, 3}]
```

Out[165]=

$$\left\{\omega \to 0, \omega \to \sqrt{\frac{k}{m_1}}, \omega \to \sqrt{k\left(\frac{1}{m_1} + \frac{2}{m_2}\right)}\right\}$$

In[166]:=

```
Table[Solve[eq3 /. freqRule[[]] /. η1 → 1, {η2, η3}], {i, 1, 3}]
```

Out[166]=

$$\{\{\{\eta_3 \to 1, \eta_2 \to 1\}\}, \{\{\eta_3 \to 1, \eta_2 \to 1\}\}, \{\{\eta_3 \to 1, \eta_2 \to 1\}\}\}$$

The nature of these solutions should now be obvious. The zero-frequency mode represents the motion of the entire system together without any internal vibration. In the second mode the central mass remains stationary while the two outer mass vibrate in opposite directions symmetrically. For the third mode the two outer masses move together while the central mass moves in the opposite direction, with the relative amplitudes proportional to the ratio of masses.

There is a more elegant way to obtain the eigenvalues and eigenvectors. We can transform the system of equations to the form $h.\eta == -\omega^2 \eta$ and then use **Eigensystem** to obtain both the eigenvalues $\lambda = -\omega^2$ and the corresponding eigenvectors directly. First we extract the right-hand sides of **eq2** and divide out the masses.

In[167]:=

```
rhs = #[[2]] & /@ eq2 /{m1, m2, m1} // Simplify
```

Out[167]=

$$\left\{\frac{k\,(-\eta_1+\eta_2)}{m_1},\ \frac{k\,(\eta_1-2\,\eta_2+\eta_3)}{m_2},\ \frac{k\,(\eta_2-\eta_3)}{m_1}\right\}$$

Then we construct the coefficient matrix.

In[168]:=

```
h = Coefficient[#, {η1, η2, η3}] & /@ rhs;
```

In[169]:=

```
h // MatrixForm
```

Out[169]//MatrixForm=

$$\begin{pmatrix} -\frac{k}{m_1} & \frac{k}{m_1} & 0 \\ \frac{k}{m_2} & -\frac{2\,k}{m_2} & \frac{k}{m_2} \\ 0 & \frac{k}{m_1} & -\frac{k}{m_1} \end{pmatrix}$$

Finally, we evaluate the eigenvalues and eigenvectors

In[170]:=

```
{λ, vectors} = Eigensystem[h]
```

Out[170]=

$$\left\{\left\{0,\ -\frac{k}{m_1},\ \frac{-2\,k\,m_1-k\,m_2}{m_1\,m_2}\right\},\ \left\{\{1,1,1\},\ \{-1,0,1\},\ \left\{1,\ -\frac{2\,m_1}{m_2},\ 1\right\}\right\}\right\}$$

and compute the frequencies.

In[171]:=

```
frequencies = Sqrt[-λ]
```

Out[171]=

$$\left\{0, \sqrt{\frac{k}{m_1}}, \sqrt{-\frac{-2\,k\,m_1 - k\,m_2}{m_1\,m_2}}\right\}$$

Note that **Eigensystem** does not normalize the eigenvectors. It is often useful to construct a matrix containing the normalized eigenvectors in column form.

In[172]:=

```
modes = vectors/(Sqrt[#.#] & /@ vectors)
```

Out[172]=

$$\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}, \left\{-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\}, \left\{\frac{1}{\sqrt{2+\frac{4\,m_1^2}{m_2^2}}}, -\frac{2\,m_1}{\sqrt{2+\frac{4\,m_1^2}{m_2^2}}\;m_2}, \frac{1}{\sqrt{2+\frac{4\,m_1^2}{m_2^2}}}\right\}\right\}$$

In[173]:=

```
Transpose[modes] // MatrixForm
```

Out[173]//MatrixForm=

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2+\frac{4\,m_1^2}{m_2^2}}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2\,m_1}{\sqrt{2+\frac{4\,m_1^2}{m_2^2}}\;m_2} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2+\frac{4\,m_1^2}{m_2^2}}} \end{pmatrix}$$

Exercises

Write a function which returns the eigenfrequencies and normal modes given a coupling matrix, h, for which $h.\eta == m.\ddot{\eta} == -\omega^2 m.\eta$ where η is a coordinate vector and m is a diagonal matrix carrying the masses.

Find the eigenfrequencies and normal modes assuming $\{k_{1,2} = k_{2,3} = k,\ k_{1,3=0},\ m_1 = m_3\}$. Verify that the previous results are recovered. Plot the frequency and ratio of amplitudes for the mode in which the central mass vibrates against the outer masses as a function of the ratio of masses.

Find the eigenfrequencies and normal modes assuming $\{k_{1,2} = k_{2,3} = k_1,\ m_1 = m_3\}$ and including the interaction $k_{1,3} = k_2$ between the outer masses. Verify that the previous results are recovered when the spring constant for the new force is set to zero. What is the effect of k_2?

Generalize to the case of three arbitrary masses. You will probably find that symbolic expressions for the eigenvalues are too lengthy to display comfortably and are difficult to simplify. Nevertheless, you can still plot interesting quantities. For example, plot the two nonzero frequencies together as functions of the mass ratios (either in 3D or in 2D as functions of one ratio for several values of the other).